Differential Topology

Lectures by John Milnor, Princeton University, Fall term 1958

Notes by James Munkres

Differential topology may be defined as the study of those properties of differentiable manifolds which are invariant under diffeomorphism (differentiable homeomorphism). Typical problem falling under this heading are the following:

1) Given two differentiable manifolds, under what conditions are they diffeomorphic?
2) Given a differentiable manifold, is it the boundary of some differentiable manifold-with-boundary?
3) Given a differentiable manifold, is it parallelisable?

All these problems concern more than the topology of the manifold, yet they do not belong to differential geometry, which usually assumes additional structure (e.g., a connection or a metric).

The most powerful tools in this subject have been derived from the methods of algebraic topology. In particular, the theory of characteristic classes is crucial, where-by one passes from the manifold *M* to its tangent bundle, and thence to a cohomology class in *M* which depends on this bundle.

These notes are intended as an introduction to the subject; we will go as far as possible without bringing in algebraic topology. Our two main goals are

a) Whitney's theorem that a differentiable n-manifold can be embedded as a closed subset of the euclidean space $\mathbb{R}^{2n+1}$ (see §1.32); and
b) Thom's theorem that the non-orientable cobordism group $\mathcal{N}^n$ is isomorphic to a certain stable homotopy groups (see §3.15).

Chapter I is mainly concerned with approximation theorems. First the basic definitions are given and the inverse function theorem is exploited. (§1.1 – 1.12). Next two local approximation theorems are proved, showing that a given map can be approximated by one of maximal rank. (§1.13 – 1.21). Finally locally finite coverings are used to derive the corresponding global theorems: namely Whitney's embedding theorem and Thom's transversality lemma (§1.35).

Chapter II is an introduction to the theory of vector space bundles, with emphasis on the tangent bundle of a manifold.

Chapter III makes use of the preceding material in order to study the cobordism group $\mathcal{N}^n$.

Chapter I *Embeddings and Immersions of Manifolds*

Notation. If x is in the euclidean space $\mathbb{R}^n$, the coordinate of x are denoted by $(x^1, \ldots, x^n)$. Let $||x|| = \max |x^i|$; let $C^n(r)$ denote the set of x such that $||x|| < r$; and $C^n(x_0, r)$ the set of x such that $||x - x_0|| < r$. The closure of a cube C is denoted by $\overline{C}$.

A real valued function $f(x^1, \ldots, x^n)$ is ***differentiable*** if the partials of f of all orders exist and are continuous (i.e., "differentiable" means C^∞). A map $f = (f^1, \ldots, f^p) : U \to \mathbb{R}^p$ (where U is an open set, in $\mathbb{R}^n$) is differentiable if each of the coordinate functions $f^1, \ldots, f^p$ is differentiable. Df denotes the Jacobian matrix of f; one verifies that $D(gf) = Dg{\cdot}Df$. The notation $\partial(f^1, \ldots, f^p)/\partial(x^1, \ldots, x^n)$ is also used. If $n = p$, $|Df|$ denotes the determinant.

1.1 ***Definition.*** A ***topological n-manifold*** M^n is a Hausdorff space with a countable basis which is locally homeomorphic to $\mathbb{R}^n$.

A ***differentiable structure*** $\mathcal{D}$ on a topological manifold M^n is a collection of real-valued functions, each defined on an open subset of M^n such that:

1) For every point p of M^n there is a neighbourhood U of p and a homeomorphism h of U onto an open subset of $\mathbb{R}^n$ such that a function f, defined on the open subset W of U, is in $\mathcal{D}$ if and only if fh^{-1} is differentiable.
2) If U_i are open sets contained in the domain of f and $U = \cup U_i$, then $f \mid U \in \mathcal{D}$ if and only if $f \mid U_i$ is in $\mathcal{D}$, for each i.

A ***differentiable manifold*** M^n is a topological manifold provided with a differentiable structure $\mathcal{D}$; the elements of $\mathcal{D}$ are called the ***differentiable functions*** on M^n. Any open set U and homeomorphism h which satisfy the requirement of 1) above are called a ***coordinate system*** on M^n.

Notation. A coordinate system is sometimes denoted by the coordinate functions: $h(p) = (u^1(p), \ldots, u^n(p))$.

1.2 ***Alternate definition.*** Let a collection (U_i, h_i) be given, where h_i is a homeomorphism of the open subset U_i of M^n onto an open subset of $\mathbb{R}^n$, such that

a) the U_i's cover M^n;
b) $h_j h_i^{-1}$ is a differentiable map on $h_i(U_i \cap U_j)$, for all i, j.

Define a ***coordinate system*** as an open set U and homeomorphism h of U onto an open subset of $\mathbb{R}^n$ such that $h_i h^{-1}$ and hh_i^{-1} are differentiable on $h(U \cap U_i)$ and $h_i(U \cap U_i)$ respectively, for each i.

Define a ***differentiable structure*** on M^n as the collection of all such coordinate systems. A function f, defined on the open set V, is ***differentiable*** if fh^{-1} is differentiable on $h(U \cap V)$, for all coordinate systems (U, h).

One shows readily that these two definitions are entirely equivalent.

1.3 ***Definition.*** Let M_1, M_2 be differentiable manifolds. If U is an open subset of M_1, $f : U \to M_2$ is ***differentiable*** if for every differentiable function g on M_2, gf is differentiable on M_1.

If $A \subset M_1$, a function $f : A \to M_2$ is ***differentiable*** if it can be extended to a differentiable function defined on a neighbourhood U of A.

$f : M_1 \to M_2$ is a ***diffeomorphism*** if f and f^{-1} are defined and differentiable.
(A coordinate system (U, h) on M^n is then an open set U in M^n and a diffeomorphism h of U onto an open set in $\mathbb{R}^n$.)

If $A \subset M$, we have just defined the notion of differentiable function for subsets of A. Suppose that A is locally diffeomorphic to $\mathbb{R}^k$: this collection is easily shown to be a differentiable structure on A. In this case, A is said to be a ***differentiable submanifold*** of M.

The following lemma is familiar from elementary calculus.

1.4. *Lemma.* *Let $f : C^n(r) \to \mathbb{R}^n$ satisfy the condition $|\partial f^i/\partial x^j| \leq b$ for all i, j. Then $\|f(x) - f(y)\| \leq bn\|(x - y)\|$, for all $x, y \in \overline{C}^n(r)$.*

1.5. *Theorem* (Inverse Function Theorem). *Let U be an open subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}^n$ be differentiable, and let Df be non-singular at x_0. Then f is a diffeomorphism of some neighbourhood of x_0 onto some neighbourhood of $f(x_0)$.*

***Proof*:** We may assume $x_0 = f(x_0) = 0$, and that $Df(x_0)$ is the identity matrix.
Let $g(x) = f(x) - x$, so that $Dg(0)$ is the zero matrix. Choose $r > 0$ so that $x \in U$ and $Df(x)$ is non-singular and $|\partial g^i/\partial x^j)| \leq 1/2n$, for all x with $\|x\| < r$.

Assertion. If $y \in C^n(r/2)$, there is exactly one $x \in C^n(r)$ such that $f(x) = y$:

By the previous lemma,

$$\|g(x) - g(x_0)\| \leq (½)\|x - x_0\| \text{ on } C^n(r). \tag{*}$$

Let us define $\{x_n\}$ inductively by $x_0 = 0$, $x_1 = y$, $x_{n+1} = y - g(x_n)$. This is well-defined, since $x_n - x_{n-1} = g(x_{n-2}) - g(x_{n-1})$ so that

$$\|x_n - x_{n-1}\| \leq (½)\|x_{n-2} - x_{n-1}\| \leq \|y\|/2^{n-1};$$

and thus $\|x_n\| \leq 2\|y\|$ for each n. Hence the sequence $\{x_n\}$ converges to a point x with $\|x\| \leq 2\|y\|$, so that $x \in C^n(r)$. Then $x = y - g(x)$, so that $f(x) = y$. This proves the existence of x. To show uniqueness, note that if $f(x) = f(x_1) = y$, then $g(x_1) - g(x) = x - x_1$, contradicting (*).
Hence $f^{-1} : C^n(r/2) \to C^n(r)$ exists. Note that

$$\|f(x) - f(x_1)\| \geq \|x - x_1\| - \|g(x) - g(x_1)\| \geq (½)\|x - x_1\|$$

so that $\|y - y_1\| \geq (½)\|f^{-1}(y) - f^{-1}(y_1)\|$. Hence f^{-1} is continuous; the image $C^n(r/2)$ of under f^{-1} is open because it equals $C^n(r) \cap f^{-1}(C^n(r/2))$, the intersection of two open sets.
To show that f^{-1} is differentiable, note that

$$f(x) = f(x_1) + Df(x_1)\cdot(x - x_1) + h(x, x_1),$$

where $(x - x_1)$ is written as a column matrix and the dot stands for matrix multiplication. Here $h(x, x_1) / \|x - x_1\| \to 0$ as $x \to x_1$. Let A be the inverse matrix of $Df(x_1)$. Then

$$A\cdot(f(x) - f(x_1)) = (x - x_1) + A\cdot h(x, x_1), \quad \text{or}$$
$$A\cdot(y - y_1) + A\cdot h_1(y, y_1) = f^{-1}(y) - f^{-1}(y_1),$$

where $h(y, y_1) = -h(f^{-1}(y), f^{-1}(y_1))$. Now

$$h_1(y, y_1) / \|y - y_1\| = -[h(x, x_1) / \|x - x_1\|](\|x - x_1\| / \|y - y_1\|).$$

Since $\|x - x_1\| / \|y - y_1\| \leq 2$, $h_1(y, y_1) / \|y - y_1\| \to 0$ as $y \to y_1$. Hence

$$D(f^{-1}) = A = (Df)^{-1}.$$

This means that $(Df)^{-1}$ is obtained as the composition of the following maps:

$$C^n(r/2) \xrightarrow[f^{-1}]{} C^n(r) \xrightarrow[Df]{} \mathrm{GL}(n) \xrightarrow[\text{matrix inversion}]{} \mathrm{GL}(n);$$

where $\mathrm{GL}(n)$ denotes the set of non-singular $n \times n$ matrices, considered as a subspace of n^2-dimensional euclidean space. Since f^{-1} is continuous and Df and matrix inversion are C^∞, $(Df)^{-1}$ is continuous, i.e., is f^{-1} is C^1. In general, if f^{-1} is C^k, then by this argument $(Df)^{-1}$ is also, i.e., f^{-1} is of class C^{k+1}. This completes the proof. □

1.6. *Lemma.* *Let U be an open subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}^p$ $(n \le p)$, $f(0) = 0$, and let $Df(0)$ have rank n. Then there exists a diffeomorphism g of one neighbourhood of the origin in $\mathbb{R}^p$ onto another so that $g(0) = 0$ and $gf(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$, in some neighbourhood of the origin.*

***Proof*:** Since $\partial(f^1, \ldots, f^p)/\partial(x^1, \ldots, x^n)$ has rank n, we may assume that

$$\partial(f^1, \ldots, f^n)/\partial(x^1, \ldots, x^n)$$

is the submatrix which is non-singular. Define $F : U \times \mathbb{R}^{p-n} \to \mathbb{R}^p$ by the equation

$$F(x^1, \ldots, x^p) = f(x^1, \ldots, x^n) + (0, \ldots, 0, x^{n+1}, \ldots, x^p).$$

F is an extension of f, since $F(x^1, \ldots, x^n, 0, \ldots, 0) = f(x^1, \ldots, x^n)$.
DF is non-singular at the origin, since its determinant everywhere equals

$$|\partial(f^1, \ldots, f^n)/\partial(x^1, \ldots, x^n)|,$$

which is non-zero. Hence F has a local inverse g, so that g maps one neighbourhood of the origin in $\mathbb{R}^p$ onto another, and

$$gF(x^1, \ldots, x^p) = (x^1, \ldots, x^p)$$

and hence

$$gf(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0).$$ □

1.7. *Corollary.* *Let A^k be a differentiable sub-manifold of M^n. Given $x \in A^k$, there is a coordinate system (U, h) on M^n about x, such that $h(U \cap A) = h(U) \cap \mathbb{R}^k$ (where $\mathbb{R}^k$ is considered as the subspace $\mathbb{R}^k \times 0$ of $\mathbb{R}^k \times \mathbb{R}^k = \mathbb{R}^n$).*

***Proof*:** Let (U_i, h_i) be a coordinate system on M^n about x; by hypothesis, there is a differentiable map f of a neighbourhood V of x in M^n into $\mathbb{R}^k$ such that $f \mid V \cap A = f_1$ is a diffeomorphism whose range is an open set W in $\mathbb{R}^k$. We may assume $U_1 = V$, and $h_1(x) = f(x) = 0$.
Now $fh_1^{-1}h_1f^{-1}$ is the identity on W, so that its Jacobian, which equals $D(fh_1^{-1})$, $D(h_1f^{-1})$ is non-singular. Hence $D(h_1f^{-1})$ has rank k, so that by the previous lemma, there is a diffeomorphism g of some neighbourhood $V_1 \subset h_1(U_1)$ of 0 onto another such that $g(0) = 0$ and $gh_1f_1^{-1}(x^1, \ldots, x^k) = (x^1, \ldots, x^k, 0, \ldots, 0)$. Then $U = h_1^{-1}(V_1)$ and $h = gh_1$ will satisfy the requirement of the lemma. □

1**.8. *Lemma.*** *Let U be an open subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}^p$, $f(0) = 0$, $(n \ge p)$, and let $Df(0)$ have rank p. Then there is a diffeomorphism h of some neighbourhood of the origin in $\mathbb{R}^n$ onto another such that $h(0) = 0$ and $fh(x^1, \ldots, x^n) = (x^1, \ldots, x^p)$.*

***Proof*:** We nay assume $\partial(f^1, \ldots, f^p)/\partial(x^1, \ldots, x^p)$ is non-singular at 0, since $Df(0)$ has rank p. Define $F : U \to \mathbb{R}^n$ by the equation

$$F(x^1, \dots, x^n) = (f^1(x), \dots, f^p(x), x^{p+1}, \dots, x^p).$$

Then $DF(0)$ is non-singular; let h be the local inverse of F. Let g project $\mathbb{R}^n$ onto the subspace $\mathbb{R}^p$; $f = gF$. Then

$$fh(x^1, \dots, x^n) = gFh(x^1, \dots, x^n) = g(x^1, \dots, x^n) = (x^1, \dots, x^p). \qquad \square$$

1.9. *Exercise.* Let U be an open subset of $\mathbb{R}^n$, $f: U \to \mathbb{R}^p$, $f(0) = 0$; and let $Df(x)$ have rank k for all x in U. Then there are local diffeomorphisms h and g of $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively such that

$$gfh(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0).$$

1.10. *Definition.* If $f: M_1 \to M_2$, the ***rank*** of f, written rank(f), at x is the rank of $D(h_2 f h_1^{-1})$ at $h_1(x)$, where (U_1, h_1) and (U_2, h_2) are coordinate systems about x and $f(x)$, respectively. The differentiable map $f: M_1^n \to M_2^p$ is an ***immersion*** if rank(f) = n everywhere ($n \le p$). It is an ***embedding*** if it is also a homeomorphism into.

If $f: M_1^n \to M_2^p$, then $y \in M_2^p$ is a ***regular value*** of f if rank(f) $= p$ on the entire set $f^{-1}(y)$. Otherwise, y is a ***critical value***. (If $y \notin f(M_1^n)$, y is, by definition, a regular value of f.)

1.11. *Exercise.* If A is a differentiable submanifold of M, the inclusion $A \to M$ is an embedding and conversely if $f: M_1 \to M$ is an embedding then $f(M_1)$ is a differentiable submanifold .

1.12. *Exercise.* If y is a regular value of $f: M_1^n \to M_2^p$, then $f^{-1}(y)$ is a differentiable submanifold of M_1^n of dimension $n - p$ (or empty).

1.13. *Definition.* A subset A of $\mathbb{R}^n$ has ***measure zero*** if it may be covered by a countable collection of cubes $C^n(x, r)$ having arbitrarily small total volume. In such a case, $\mathbb{R}^n \setminus A$ is everywhere dense (i.e., it intersects every non-empty open set).

1.14. *Lemma.* *Let U be an open subset of* $\mathbb{R}^n$; let $f: U \to \mathbb{R}^n$ *be differentiable. If $A \subset U$ has measure zero, so does $f(A)$.*

***Proof*:** Let C be any cube with $\overline{C} \subset U$. Let b denote the maximum of $|\partial f^i/\partial x^j|$ on $\overline{C}$ for all i, j. By 1.4, $\|f(x) - f(y)\| \le bn\|x - y\|$ for $x, y \in \overline{C}$.
Now $A \cap C$ has measure zero; let us cover $A \cap C$ by cubes $C(x_i, r_i)$ with closure contained in C, such that $\sum_{i=1,\infty} r_i^n < \varepsilon$. Then $f(C(x_i, r_i)) \subset C(f(x_i), bnr_i)$, so that $f(A \cap C)$ is covered by cubes of total volume $b^n n^n \sum_{i=1,\infty} r_i^n < b^n n^n \varepsilon$. Hence $f(A \cap C)$ has measure zero.
Since A can be covered by countably many such cubes C, $f(A)$ has measure zero. $\square$

1.15. *Corollary.* If $f: U \to \mathbb{R}^n$ *be differentiable, where U is an open subset of* $\mathbb{R}^n$ *and $n < p$, then $f(U)$ has measure zero.*

***Proof*:** Project $U \times \mathbb{R}^{p-n}$ onto U and apply f. Since $U \times 0$ has measure zero in $\mathbb{R}^p$, so does $f(U)$. $\square$

1.16. *Definition.* If $A \subset M$, M has ***measure zero*** if $h(A \cap U)$ has measure zero for every coordinate system (U, h).

1.17. *Corollary.* If $f: M_1^n \to M_2^p$ *is differentiable and $n < p$, then $f(M_1^n)$ has measure zero.*

1.18. *Definition.* Let $\mathcal{M}(p, n)$ denote the space of $p \times n$ matrices, with the differentiable structure of the euclidean space $\mathbb{R}^{pn}$. Let $\mathcal{M}(p, n; k)$ denote the subspace consisting of matrices of rank k. Thus $\mathcal{M}(p, n; n)$ is an open subset of $\mathcal{M}(p, n)$ if $p \ge n$; the determinantal criterion for rank proves this. More generally, we have:

1.19. *Lemma.* *$\mathcal{M}(p, n; k)$ is a differentiable submanifold of $\mathcal{M}(p, n)$ of dimension $k(p + n - k)$, where $k \le \min(p, n)$.*

Proof: Let $E_0 \in \mathcal{M}(p, n; k)$; we may assume that E_0 is of the form, $\begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}$, where A_0 is a non-singular $k \times k$ matrix. There is an $\varepsilon > 0$ such that if all the entries of $A - A_0$ are less than ε, A must also be non-singular. Let U consist of all matrices in $M(p, n)$ of the form $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, with all the entries of $A - A_0$ are less than ε.

Then E is in $\mathcal{M}(p, n; k)$ if and only if $D = CA^{-1}B$: for the matrix

$$\begin{bmatrix} I_k & 0 \\ X & I_{p-k} \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ XA + C & XB + D \end{bmatrix}$$

has the same rank as E. If $X = -CA^{-1}$, this matrix is

$$\begin{bmatrix} A & B \\ 0 & CA^{-1}B + D \end{bmatrix}.$$

If $D = CA^{-1}B$, this matrix has rank k. The converse also holds, for if any element of $-CA^{-1}B + D$ is different from zero, this matrix has rank $> k$.

Let W be the open set in euclidean space of dimension

$$(pn - (p - k)(n - k)) = k(p + n - k)$$

consisting of matrices $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ with all the entries of $A - A_0$ are less than ε. The map

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \rightarrow \begin{bmatrix} A & B \\ 0 & CA^{-1}B + D \end{bmatrix}$$

is then a diffeomorphism of W onto the neighbourhood $U \cap \mathcal{M}(p, n; k)$ of E_0. □

1.20. *Theorem.* *Let U be an open set in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^p$ be differentiable, where $p \ge 2n$. Given $\varepsilon > 0$, there is a $p \times n$ matrix $A = (a^i_j)$ with each $|a^i_j| < \varepsilon$, such that $g(x) = f(x) + A\cdot x$ is an immersion (x written as a column matrix.)*

Proof: $Dg(x) = Df(x) + A$; we would like to choose A in such a way that $Dg(x)$ has rank n for all x. I.e., A should be of the form $Q - Df$, where Q has rank n.

We define $F_k : \mathcal{M}(p, n; k) \times U \to \mathcal{M}(p, n)$ by the equation

$$F_k(Q, x) = Q - Df(x).$$

Now F_k is a differentiable map, and the domain of F_k has dimension $k(p + n - k) + n$. As long as $k < n$, this expression is monotonic in k (its partial derivative with respect to k is $p + n - 2k$). Hence the domain of F_k has dimension not greater than

$$(n - 1)(p + n - (n - 1)) + n = (2n - p) + pn - 1$$

for $k < n$. Since $p \ge 2n$, this dimension is strictly less than $pn = \dim(\mathcal{M}(p, n))$.

Hence the image of F_k has measure zero in $\mathcal{M}(p, n)$, so that there is an element A of $\mathcal{M}(p, n)$, arbitrarily close to the zero matrix, which is not in the image of F_k for $k = 0, \dots, n - 1$. Then $A + Df(x) = Dg(x)$ has rank n, for each x. □

1.21. ***Theorem.*** *Let U be an open subset of $\mathbb{R}^n$; and let $f : U \to \mathbb{R}^p$ be differentiable. Given $\varepsilon > 0$, there are matrices $A(p \times n)$ and $B(p \times 1)$ with entries less than ε in absolute value such that*

$$g(x) = f(x) + A \cdot x + B$$

has the origin as a regular value.

Remark. The following much more delicate result has been proved by [Sard, A.]: *The set of critical values of any differentiable map has measure zero.*

***Proof of* 1.21.** Note that the theorem is trivial if $p > n$, since then $f(U)$ has measure zero, and we may choose $A = 0$ and B small in such a way that 0 is not in the image of g.
Assume $p \le n$. We wish $Dg(x_0) = Df(x_0) + A$ to have rank p, where x_0 ranges over all points such that

$$g(x_0) = 0 = f(x_0) + A \cdot x_0 + B.$$

Hence A is of the form $Q - Df(x)$, and B is of the form $-f(x) - A \cdot x$, where Q is to have rank p.
We define $F_k : \mathcal{M}(p, n; k) \times U \to \mathcal{M}(p, n) \times \mathbb{R}^p$ by the equation

$$F_k(Q, x) = (Q - Df(x), -f(x) - (Q - Df(x)) \cdot x).$$

Then F_k is differentiable. If $k < p$, the dimension of its domain is not greater than $(p - 1)((p + n - (n - 1)) + n = p + pn - 1$. Hence the image of F_k, $k = 0, \dots, p - 1$ has measure zero; so that there is a point (A, B) arbitrarily close to to the origin which is not in any such image set. This completes the proof. □

1.22. ***Definition.*** A covering of a topological space X is ***locally-finite*** if every point has a neighbourhood which intersects only finitely many elements of the covering. A ***refinement*** of a covering of X is a second covering each element of which is contained in an element of the first covering. A Hausdorff space is ***paracompact*** if every open covering has a locally-finite open refinement.

If X is paracompact, and $\{U_\alpha\}$ is an open covering, there is a locally-finite open covering $\{V_\alpha\}$ with $V_\alpha \subset U_\alpha$ for each α. For let $\{W_\beta\}$ be a locally-finite refinement of $\{U_\alpha\}$; choose $\alpha(\beta)$ so that $W_\beta \subset U_{\alpha(\beta)}$ for each β. Set $V_{\alpha 0} = U_{\alpha(\beta) = \alpha 0} W_\beta$. Given a neighbourhood intersecting only finitely many W_β, it intersects only finitely many V_α as well.

1.23. ***Theorem.*** *A locally compact Hausdorff space having a countable basis is paracompact.*

***Proof*:** Let X be paracompact and let $U_1, U_2, \dots$ be a basis for X with $\overline{U}_i$ compact with each i. There exists a sequence $A_1, A_2, \dots$ of compact sets whose union is X, such that $A_i \subset \operatorname{Int} A_{i+1}$: set $A_1 = \overline{U}_1$. Given A_i compact, let k be the smallest integer such that A_i is contained in $U_1 \cup \dots \cup U_k$; Let A_{i+1} equal the closure of this set union $\overline{U}_{i+1}$.
Let O be an open covering of X. Cover the compact set $A_{i+1} \setminus \operatorname{Int} A_i$ by a finite number of open sets $V_1, \dots V_n$ where each V_i is contained in some element of O, and in the open set $\operatorname{Int} A_{i+2} \setminus A_{i-1}$. Let P_i denote the collection $\{V_1, \dots V_n\}$, and let $P = P_0 \cup P_1 \cup \dots$. P refines O, and since any compact closed neighbourhood C is contained in some A_i, C can intersect only finitely many elements of P. □

1.24. ***Exercise.*** Prove that a paracompact space is normal. (First prove that it is regular.)

1.25. *Theorem.* *Let M^n be a differentiable manifold, $\{U_\alpha\}$ an open covering of M^n. There is a collection (V_j, h_j) of coordinate systems on M^n such that*

1) $\{V_j\}$ is a locally-finite refinement of $\{U_\alpha\}$.
2) $h_j(V_j) = C^n(3)$.
3) If $W_j = h_j^{-1}((C^n(1))$, then $\{W_j\}$ covers M^n.

***Proof*:** The proof proceeds along lines similar to the previous one. The only difference is that one chooses the V_j to satisfy 2), and makes sure that the sets $h_j^{-1}((C^n(1))$ also cover $A_{i+1} \setminus \mathrm{Int}A_i$. □

1.26. We wish to construct a C^∞ function $\varphi(x^1, \ldots, x^n)$ such that $\varphi = 1$ on $\overline{C}^n(1)$, $0 < \varphi < 1$ on $C^n(2) \setminus \overline{C}^n(1)$, $\varphi = 0$ on $\mathbb{R}^n \setminus C^n(2)$.
This function may be defined by the equation $\varphi(x^1, \ldots, x^n) = \prod_{i=1,n}\psi(x^i)$, where

$$\psi(x) = \lambda(2 + x)\cdot\lambda(2 - x) / [\lambda(2 + x)\cdot\lambda(2 - x) + \lambda(x - 1) + \lambda(-x - 1)]$$

and

$$\lambda(x) = \begin{matrix} \exp(-1/x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{matrix}$$

Note that the denominator in the expression for ψ is always positive, and that

$$\begin{aligned} \psi(x) = 1 \quad & \text{for} \quad |x| \leq 1 \\ 0 < \psi(x) < 1 \quad & \text{if} \quad 1 < |x| < 2 \\ \psi(x) = 0 \quad & \text{if} \quad |x| \geq 2. \end{aligned}$$

1.27. *Definition.* Let $f, g : X \to Y$, where Y is metrisable, and let $\delta(x)$ be a positive continuous function defined on X. Then g is a ***δ-approximation to*** f if $d(f(x), g(x)) < \delta(x)$ for all x. [If one takes the δ-approximation to f to be a neighbourhood of f in the function space $F(X, Y)$, this imposes a topology on the function space, independent of the metric on Y provided X, Y are paracompact.]

1.28. *Theorem.* *Given a differentiable map $f : M^n \to \mathbb{R}^p$ where $p \geq 2n$, and a continuous positive function δ on M^n, there exists an immersion $g : M^n \to \mathbb{R}^p$ which is a δ-approximation to f. If rank $f = n$ on the closed set N, we may choose $g \mid N = f \mid N$.*

***Proof*:** Note that rank $f = n$ on a neighbourhood U of N. Cover M^n by U and $M^n \setminus N$. Let (V_j, h_j) be a refinement of this covering, constructed as in 1.25. As before, $h_i(\overline{W_i}) = C^n(1)$ and $h_i(V_i) = C^n(3)$. Let $h_j(U_j) = C^n(2)$. Let the V_i be so indexed with positive and negative integers that those V_i with non-positive indices are the ones contained in U. Let $\varepsilon_1 = \min$ of $\delta(x)$ on the compact set $\overline{U_i}$.
Set $f_0 = f$. Given $f_{k-1} : M^n \to \mathbb{R}^p$, having rank n on $N_{k-1} = \cup_{j<k}W_i$, consider $f_{k-1}h_k^{-1} : C^n(3) \to \mathbb{R}^p$.
Let A be a $p \times n$ matrix; let $F_A : C^n(3) \to \mathbb{R}^p$ be defined by the equation

$$F_A(x) = f_{k-1}h_k^{-1}(x) + \varphi(x)A\cdot(x),$$

where (x) is written (as usual) as a column matrix $(n \times 1)$; A is yet to be chosen; and $\varphi(x)$ is the function defined in 1.26.
First, we want $F_A(x)$ to have rank n on the set $K = h_k(N_{k-1} \cap \overline{U}_k)$; we are given that $f_{k-1}h_k^{-1}$ has rank n on K. Thus

$$D(F_A(x)) = D(f_{k-1}h_k^{-1}(x)) + A\cdot(x)\cdot D\varphi(x) + \varphi(x)A.$$

($D\varphi$ is a $1 \times n$ matrix.) The map of $K \times \mathcal{M}(p, n)$ into $\mathcal{M}(p, n)$ which carries (x, A) into $D(F_A(x))$ is continuous. It carries $K \times (0)$ into the open subset $\mathcal{M}(p, n; n)$ of $\mathcal{M}(p, n)$. Hence if A is sufficiently small, this map will carry $K \times A$ into $\mathcal{M}(p, n; n)$; our first requirement is that A be this small.
Secondly, we require A to be small enough that $\|A\cdot(x)\| < \varepsilon_k/2^k$ for all $x \in C^n(3)$.
Finally, by 1.20, A may be chosen arbitrarily small so that $f_{k-1}h_k^{-1}(x) + A\cdot(x)$ has rank n on $C^n(2)$.
Let A be chosen to satisfy this requirement.
We then define $f_k : M^n \to \mathbb{R}^p$ by the equation:

$$f_k(y) = \begin{cases} f_{k-1}(y) + \varphi(h_k(y))A\cdot h_k(y) & \text{for } y \in V_k \\ f_{k-1}(y) & \text{for } y \in M \setminus \overline{U}_k. \end{cases}$$

These definitions agree on the overlapping domains, so that f_k is differentiable. By the first condition on A, it has rank n on N_{k-1}; by the third condition it has rank n on $\overline{W}_k$. By the second condition, f_k is a $\delta/2^k$ approximation to f_{k-1}.
We define $g(x) = \lim_{k\to\infty} f_k(x)$. Since the covering V_k is locally-finite, all the f_k agree on a given compact set for k sufficiently large; it follows that g is differentiable and has rank n everywhere. It is also a δ-approximation to f. □

1.29. *Lemma.* *If $p > 2n$, any immersion $f : M^n \to \mathbb{R}^p$ can be δ-approximated by a* 1 - 1 *immersion g. If f is* 1 - 1 *in a neighbourhood U of the closed set N, we may choose $g \mid N = f \mid N$.*

Proof: Choose a covering $\{U_\alpha\}$ of M^n such that $f \mid U_\alpha$ is an embedding (possible by 1.6). Let (V_i, h_i) be the locally-finite refinement constructed in 1.25; let $\varphi(x)$ be the function constructed in 1.26.
Let

$$\varphi_1(y) = \begin{cases} \varphi(h_1(y)) & \text{for } y \in V_i \\ 0 & \text{for other } y. \end{cases}$$

Then φ_1 is differentiable. As before, we assume (V_i, h_i) refines the covering $(U, M^n \setminus N)$ and that those V_i with non-positive indices are the ones contained in U.
Let $f_0 = f$. Given the immersion $f_{k-1} : M^n \to \mathbb{R}^p$, we define f_k by the equation

$$f_k(y) = f_{k-1}(y) + \varphi_k(y)b_k,$$

where b_k is a point of $\mathbb{R}^p$ yet to be chosen. By the argument of the previous theorem, if b_k is chosen sufficiently small, f_k will have rank n everywhere. The first requirement is that b_k be this small; the second requirement is that b_k be small enough that f_k be a $\delta/2^k$ approximation to f_{k-1}.
Finally, let N^{2n} be the open subset of $M^n \times M^n$ consisting of pairs (y, y_0), with $\varphi_k(y) \neq \varphi_k(y_0)$.
Consider the differentiable map

$$(y, y_0) \mapsto -[f_{k-1}(y) - f_{k-1}(y_0)] \,/\, [\varphi_k(y) - \varphi_k(y_0)]$$

from N^{2n} into $\mathbb{R}^p$. Since $2n < p$, the image of N^{2n} has measure zero, so that b_k may be chosen arbitrarily small and ***not*** in this image. It follows that $f_k(y) = f_k(y_0)$ if and only if $\varphi_k(y) = \varphi_k(y_0)$ and $f_{k-1}(y) = f_{k-1}(y_0)$ $(k > 0)$.
Define $g(y) = \lim_{k\to\infty} f_k(y)$. If $g(y) = g(y_0) =$ and $y \neq y_0$, it would follow that $f_{k-1}(y) = f_{k-1}(y_0)$ and $\varphi_k(y) = \varphi_k(y_0)$ for all $k > 0$. The former condition implies that $f(y) = f(y_0)$, so that y and $= y_0$ cannot belong to any one set U_i. Because of the latter condition, this means that neither is in any set U_i for $i > 0$. Hence, they lie in U, contradicting the fact that f is 1 - 1 on U. □

1.30. *Definition*. Let $f : M^n \to \mathbb{R}^p$. The ***limit set*** $L(f)$ is the set of $y \in \mathbb{R}^p$ such that $y = \lim f(x_n)$ for some sequence $\{x_1, x_2, \ldots\}$ which has no limit point on M^n.

Exercise. Show the following:

1) $f(M^n)$ is a closed subset of $\mathbb{R}^p$ if and only if $L(f) \subset f(M^n)$
2) f is a topological embedding if and only if f is 1 - 1 and $L(f) \cap f(M^n)$ is vacuous.

1.31. ***Lemma.*** *There exists a differentiable map $f : M^n \to \mathbb{R}$ with $L(f)$ empty.*

***Proof*:** Let (V_i, h_i) and φ be chosen as in 1.25 and 1.26 with i ranging over positive integers; let

$$\varphi_i(y) = \begin{cases} \varphi(h_i(y)) & \text{if } y \in V_i \\ 0 & \text{otherwise.} \end{cases}$$

Define $f(y) = \sum_j (j\varphi_j(y))$. This sum is finite, since V_i is a locally-finite covering. If $\{x_i\}$ is a set of points of M^n having no limit point, only finitely many lie in any compact subset of M^n. Given m, there is an integer i such that x_i is not in $\overline{W}_1 \cup \ldots \cup \overline{W}_m$. Hence $x_i \in \overline{W}_j$ for some $j > m$, whence $f(x_i) > m$. Thus the sequence $f(x_m)$ cannot converge. □

1.32. ***Corollary.*** *Every M^n can be differentiably embedded in $\mathbb{R}^{2n+1}$ as a closed subset.*

***Proof*:** Let $f : M^n \to \mathbb{R} \subset \mathbb{R}^{2n+1}$ differentiably, with $L(f) = 0$. Set $\delta(x) \equiv 1$, and let g be a 1 - 1 immersion which is a δ-approximation to f. Then $L(g)$ is empty, so that g is a homeomorphism.

1.33. ***Definition.*** Let $f : M^n \to N^p$ be differentiable. Let N_1^{p-q} be a differentiable submanifold of N^p. Let $f(x) \in N_1^{p-q}$. Let $(u^1, \ldots, u^n)$ be a coordinate system about x; and let $(v^1, \ldots, v^p)$ be a coordinate system about $f(x)$ such that on N_1^{p-q}, $v^1 = \cdots = v^p = 0$ (see 1.6). Consider the condition that $\partial(v^1, \ldots, v^q)/\partial(u^1, \ldots, u^n)$ has rank q at x. This is the ***transverse regularity condition for f and N_1^{p-q} at*** x. [Exercise: Show that this condition is independent of coordinate system.]

Note that the set of points on which the transverse regularity condition is satisfied is an open subset of $f^{-1}(N_1^{p-q})$; f is said to be ***transverse regular on*** N_1^{p-q} if the condition is satisfied foe each x in $f^{-1}(N_1^{p-q})$.

1.34. ***Lemma.*** *If $f : M^n \to N^p$ is transverse regular on N_1^{p-q} then $f^{-1}(N_1^{p-q})$ is a differentiable submanifold of dimension $n - q$ (or is empty).*

***Proof*:** Let π project $\mathbb{R}^p$ onto its first q components; $\pi : \mathbb{R}^p \to \mathbb{R}^q$. If $(V, h) = (v^1, \ldots, v^p)$ is the coordinate system hypothesised in 1.33, then

$$N_1^{p-q} \cap V = h^{-1}\pi^{-1}(0)$$

where 0 denotes the origin in $\mathbb{R}^q$; and $f^{-1}(N_1^{p-q} \cap V) = (\pi h f)^{-1}(0)$. Since $\pi h f$ has rank q at $x \in f^{-1}(N_1^{p-q} \cap V)$, the origin is a regular value of $\pi h f$. Hence $(\pi h f)^{-1}(0)$ is a differentiable submanifold of M^n of dim $n - q$ (see 1.12). □

1.35. ***Theorem.*** *Let $f : M^n \to N^p$ be differentiable; let N_1^{p-q} be a closed subset of M^n such that the transverse regularity condition for f and N_1^{p-q} holds at each x in $A \cap f^{-1}(N_1^{p-q})$. Let δ be a positive continuous function on M^n. There exists a differentiable map $g : M^n \to N^p$ such that*

1) *g is a δ-approximation to f,*
2) *g is transverse regular on N_1^{p-q}, and*

3) $g \mid A = f \mid A$.

Proof: There is a neighbourhood U of A in M^n such that f satisfies the transverse regularity condition on $U \cap f^{-1}(N_1^{p-q})$. Cover N^p by $N^p \setminus N_1^{p-q} = Y_0$ and coordinate system (Y_i, η_i) for $i > 0$; with coordinate functions $(v^1, \ldots, v^n)$ such that $v^1 = \cdots = v^p = 0$ on N_1^{p-q}. Now the open sets $f^{-1}(Y_i)$ cover M^n, as do the open sets U, $M^n \setminus A$. Let (V_j, h_j) be a refinement of both coverings, constructed as in 1.25. Recall that $h_j(V_j) = C^n(3)$, $h_j(U_j) = C^n(2)$, $h_j(W_j) = C^n(1)$, and the W_j cover M^n. The V_j are to be indexed with positive and negative integers so that those V_j which are contained in U are the ones with non-positive indices.
Let φ be as in 1.26, and define

$$\varphi_i(x) = \begin{cases} \varphi(h_i(x)) & \text{for } x \in V_i \text{ and} \\ 0 & \text{elsewhere.} \end{cases}$$

For each j choose $i(j) \geq 0$ so that $f(V_j)$ is contained in $Y_{i(j)}$.
Set $f_0 = f$. Suppose $f_{k\text{-}1}$ is defined and satisfies the transverse regularity condition for N_1^{p-q} at each point of the intersection of $f_{k\text{-}1}^{-1}(N_1^{p-q})$ with $\cup_{j<k}\overline{W}_j$. Furthermore suppose that $f_{k\text{-}1}^{-1}(\overline{U}_j) \subset Y_{i(j)}$ for each j. Setting $i = i(k)$, it follows in particular that $f_{k\text{-}1}^{-1}(\overline{U}_k) \subset Y_i$.
Consider

$$\pi\, \eta_i f_{k\text{-}1} h_k^{-1} : C^n(2) \to \mathbb{R}^q;$$

By 1.21, there is an arbitrarily small affine function $L(x) = A\cdot(x) + B$ such that when added to the previous function, the resulting map has the origin as a regular value. Consider $\mathbb{R}^q$ as the first q coordinates in $\mathbb{R}^p$, and define

$$f_k(x) = \begin{cases} \eta_i^{-1}(\eta_i f_{k\text{-}1}(x) + L(h_k(x)\, \varphi_k(x)) & \text{for } x \text{ in a neighbourhood of } \overline{U}_k \\ f_{k\text{-}1}(x) & \text{for } x \text{ in } M^n \setminus U_k. \end{cases}$$

Here L is yet to be chosen. Of course, we must choose L small enough that

$$\eta_i f_{k\text{-}1} + L\varphi_k$$

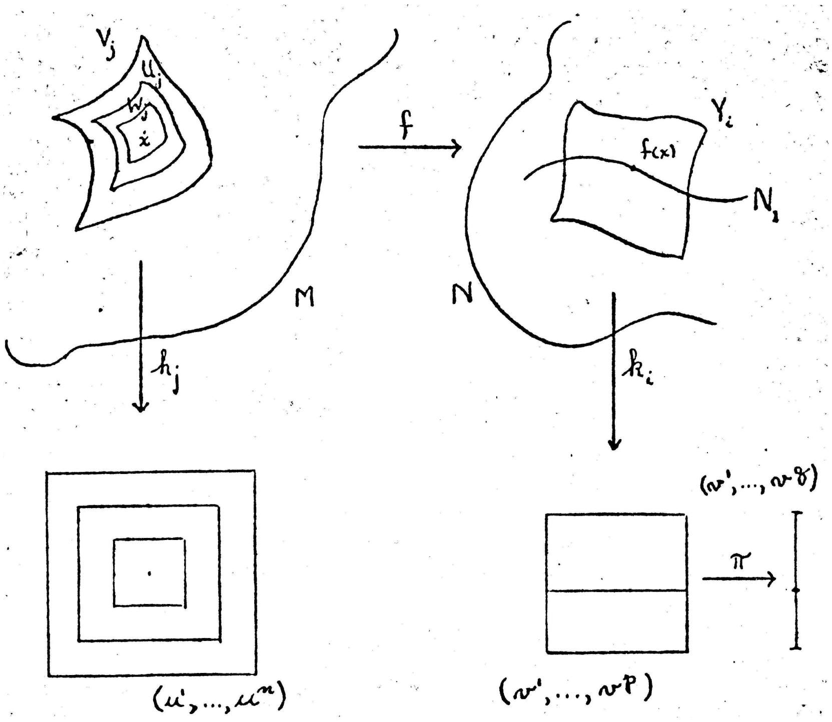
lies in $C^n(1)$ for $x \in \overline{U}_k$, in order that k_i^{-1} may be applied to it. This is the first requirement on L. Secondly, we choose L small enough that f_k is a $\delta/2^k$ approximation to $f_{k\text{-}1}$. Thirdly choose L small enough so that $f_k(\overline{U}_j)$ is contained in $Y_{i(j)}$ for each j. This is possible since only a finite number of the sets $\overline{U}_j$ can intersect $\overline{U}_k$.
Now f_k by definition satisfies the transverse regularity condition for N_1^{p-q} at each point of $f_k^{-1}(N_1^{p-q}) \cap \overline{W}_k$. We want to choose L small enough that the condition is satisfied at each point of this intersection of $f_k^{-1}(N_1^{p-q})$ with $\cup_{j<k}\overline{W}_j$. It is sufficient to consider the intersection of this set with $\overline{U}_k$; let this intersection be denoted by K. Consider the function which maps the pair (x, L) $(x \in K)$ into

$$(f_k(x), D(\pi\, \eta_i f_{k\text{-}1} h_k^{-1})\cdot(h_k(x)) \in N_1^{p-q} \times \mathcal{M}(q, n).$$

This function is continuous and carries $K \times (0)$ into the set

$$[(N^p \setminus N_1^{p-q}) \times \mathcal{M}(q, n)] \cup [N_1^{p-q} \times \mathcal{M}(q, n; q)],$$

which is open in $N_1^{p-q} \times \mathcal{M}(q, n)$. Hence for L sufficiently small, (K, L) is carried into this set, so that f_k satisfies the transverse regularity condition for N_1^{p-q} at each point of $f_k^{-1}(N_1^{p-q}) \cap (\cup_{j<k}\overline{W}_j)$.
We define $g(x) = \lim_{k\to\infty} f_k(x)$, as usual. □



V_j
U_j
W_j
x
f
Y_i
$f(x)$
N_1
M
N
h_j
k_i
$(v^1, \ldots, v^q)$
π
$(u^1, \ldots, u^n)$
$(v^1, \ldots, v^p)$

Chapter II *Vector Space Bundles*

2.1 ***Definition.*** An n-dimensional ***real vector space bundle*** ξ is a triple (π, a, s) where $\pi : E \to B$ is an onto continuous map between Hausdorff spaces that satisfy the following:

1) $F_b = \pi^{-1}(b)$, called a ***fibre,*** is an n-dimensional real vector space with $s : R \times E \to E$ carrying $R \times F_b$ into F_b, and $a : U(F_b \times F_b) \subset E \times E \to U(F_b)$ carrying $F_b \times F_b$ into F_b, as scalar product and vector addition, respectively.
2) (Local triviality) For each $b \in B$, there is a neighbourhood U of b and a homeomorphism $\varphi : U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that φ is a vector space isomorphism of $b' \times \mathbb{R}^n \cong F_{b'}$, for each $b' \in U$.

If in 2) the neighbourhood U may be taken as all B, the bundle is said to be the ***trivial bundle***.

If ξ, η are n-dimensional and p-dimensional vector space bundles, respectively, we define the ***product bundle*** $\xi \times \eta$ as follows:

$$E(\xi \times \eta) = E(\xi) \times E(\eta)$$
$$B(\xi \times \eta) = B(\xi) \times B(\eta)$$
$$(\pi \times \lambda)(x, y) = ((\pi(x), \lambda(y))$$

where π, λ are the projections in ξ, η respectively and $F_b(\xi \times \eta)$ has the usual product structures for vector spaces.

If U is a subset of $B(\xi)$, then $\xi \mid U$ denotes the bundle $\pi : \pi^{-1}(U) \to U$. It is called the ***restriction*** of the bundle to U.

2.2 ***Definition.*** Let M^n be a differentiable manifold and let x_0 be in M^n. A ***tangent vector*** at x_0 is an operation X which assigns to each differentiable function f defined in a neighbourhood U of x_0, a real number, that is, $X : \mathcal{O}(U) \to \mathbb{R}$. The following conditions must be satisfied:

1) If g is a restriction of f, $X(g) = X(f)$.
2) $X(cf + dg) = cX(f) + dX(g)$ for $c, d \in \mathbb{R}$
3) $X(f{\cdot}g) = X(f){\cdot}g(x_0) + f(x_0){\cdot}X(g)$, where the dot means ordinary real multiplication.

Then $X(1) = X(1{\cdot}1) = X(1) + X(1)$, by 3). Hence $X(1) = 0$ and $X(c)$ also $= 0$, by 2).

If one thinks of a tangent vector as being the velocity vector of a curve lying in the manifold, then $X(f)$ is merely the derivative of f with respect to the parameter of the curve. This is made more precise below.

2.3 ***Lemma.*** *Let* $(u^1, \ldots, u^n)$ *be a coordinate system about* x*. Let* X *be a tangent vector at* x*. Then* X *may be written uniquely as a linear combination of the operators* $\partial/\partial u^i$:

$$X = \textstyle\sum \alpha^i \partial/\partial u^i.$$

Proof: We assume $u(x)$ is the origin. Given any $f(u^1, \ldots, u^n)$ define

$$g(u^1, \ldots, u^n) = \begin{cases} [f(u^1, \ldots, u^n) - f(0, u^2, \ldots, u^n)] \,/\, u^1 & \text{if } u^1 \neq 0 \\ \partial f(0, u^2, \ldots, u^n)/\partial u^1 & \text{if } u^1 = 0. \end{cases}$$

To see that g is differentiable, note that

$$g(0, u^2, \ldots, u^n) = \int_{[0,1]}[\partial f(0, u^2, \ldots, u^n)/\partial u^1]dt.$$

(Then $f(u^1, \ldots, u^n) = u^1 g_1(u^1, \ldots, u^n) + f(0, u^2, \ldots, u^n)$.) Similarly,

$$f(0, u^2, \ldots, u^n) = u^2 g_2(u^2, \ldots, u^n) + f(0, 0, u^3, \ldots, u^n),$$

where $g_2(0) = \partial f/\partial u^2(0)$. Finally we have $f(u^1, \ldots, u^n) = \sum u^i g_i + f(0)$, where $g_i(0) = \partial f/\partial u^i(0)$. Thus

$$X(f) = \sum X(u^i)g_i(0) + 0\cdot X(g_i) = \sum \alpha^i \partial f/\partial u^i(0),$$

where $\alpha^i = X(u^i)$.

Remark. If $(v^1, \ldots, v^n)$ is another coordinate system about x, and $X = \sum \beta^j \partial/\partial v^j$., then $\alpha^i = X(u^i) = \sum \beta^j \partial u^i/\partial v^j$. The α^i are called the components of the vector X with respect to the coordinate system $(u^1, \ldots, u^n)$.

2.4 ***Alternate definition.*** A tangent vector at x is an assignment to every coordinate system $(u^1, \ldots, u^n)$ about x of an element $(\alpha^1, \ldots, \alpha^n)$ of $\mathbb{R}^n$, with the requirement that if (β^j) is assigned to the system $(v^1, \ldots, v^n)$, then $\alpha^i = \sum \beta^j \partial u^i/\partial v^j$. The derivation operator X is then defined as $\sum \alpha^i \partial/\partial u^i$. One checks readily that

a) $X(f)$ is independent of the coordinate system used, and

b) $X(f)$ satisfies requirements 1), 2), and 3) for a tangent vector.

2.5. ***Definition.*** For each x in M^n, the tangents at x form an n-dimensional vector space (by 2.3, the operations $\partial/\partial u^i$ form a basis). Let the totality of these be denoted by $E(\tau)$; define $\pi : E(\tau) \to M^n$ as mapping all the tangent vectors X at x_0 into x_0. The local product structure around $x_0 \in U$ is given by $\varphi_U : U \times \mathbb{R}^n \to E(\tau)$, where $(U, h) = (u^1, \ldots, u^n)$ is a coordinate system on M^n, and φ_U is defined as follows:

$$\varphi_U(x_0, a^1, \ldots, a^n) = \text{tangent vector } X = \sum \alpha^i \partial/\partial u^i \text{ at } x_0.$$

Since φ_U is to be a homeomorphism, this structure imposes a topology on $E(\tau)$; since $\varphi_V^{-1}\varphi_U$ is a homeomorphism on $(U \cap V) \times \mathbb{R}^n$, this topology is unambiguously determined. One checks immediately that φ_U gives us a vector space bundle isomorphism for each fibre.

Indeed, $\varphi_V^{-1}\varphi_U$ is a C^∞ map on $(U \cap V) \times \mathbb{R}^n$, so that $E(\tau)$ is a differentiable manifold of dimension $2n$ (using definition 1.2 of a differentiable manifold). The map π is differentiable of rank n. This bundle τ is called the ***tangent bundle*** of M^n.

2.6. ***Definition.*** If $f : M_1^n \to M_2^m$, there is an induced map $df : E(\tau_1) \to E(\tau_2)$ defined as follows: $df(X) = Y$, where $Y(g) = X(gf)$. If X is a vector at x_0, Y is a vector at $f(x_0)$. This is clearly linear on each fibre; it is called the ***derivative*** map.

If (U, h) and (V, k) are coordinate systems about $x_0, f(x_0)$ respectively, and (α^i), (β^j) are the respective components of X and Y with respect to these coordinate systems, then $(\beta^j) = D(kfh^{-1})(\alpha^i)$ where the vector components are written as column matrices, as usual.

2.7. ***Definition.*** Let ξ, η be two n-dimensional vector bundles. A ***bundle map*** $f : \xi \to \eta$ is a continuous map of $E(\xi)$ into $E(\eta)$ which carries each fibre isomorphically onto a fibre. The induced map $f_B : B(\xi) \to B(\eta)$ is automatically continuous.

If $B(\xi) = B(\eta)$ and the induced map is the identity, f is said to be an ***equivalence***. Note that if f is an equivalence, it is a homeomorphism: Locally f is just a map $U \times \mathbb{R}^n \to V \times \mathbb{R}^n$. The projection of f^{-1} into the factor U is continuous, because f_B^{-1} is the identity. But f may be given by a non-singular matrix function of $x \in U$; f^{-1} is the inverse of this matrix, so that the projection of f^{-1} into the factor $\mathbb{R}^n$ is continuous. Hence f^{-1} is continuous.

If there is an equivalence of ξ onto η, we write $\xi \simeq \eta$.

2.8. ***Lemma.*** *Given a bundle η with projection map $\lambda : E(\eta) \to B(\eta)$, and a map $f : B_1 \to B(\eta)$, there is a bundle $\pi : E_1 \to B_1$ and a bundle map $g : E_1 \to E(\eta)$ such that $\lambda g = f\pi$. Furthermore, E_1 is unique up to an equivalence.*

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E(\eta) \\ \pi\downarrow & & \downarrow\lambda \\ B_1 & \xrightarrow[f]{} & B(\eta) \end{array}$$

Remark. E_1 is called the ***induced bundle*** by f and is often denoted by $f^*\eta$.

***Proof*:** Let E_1 be that subset of $B_1 \times E(\eta)$ consisting of points (b, e) such that $f(b) = \lambda(e)$. Define $\pi(b, e) = b$; $g(b, e) = e$. To show that E_1 is a vector space bundle, let $\varphi : V \times \mathbb{R}^n \to E(\eta)$ be a product neighbourhood in $E(\eta)$, and let $f(U) \subset V$. Then define $\varphi_1 : U \times \mathbb{R}^n \to E_1$ by $\varphi_1(b, x) = (b, \varphi((b), x))$. Then φ_1 is continuous and 1 - 1; its image equals $\pi^{-1}(U)$. Its inverse φ_1^{-1} carries (b, e) into $(b, p\varphi^{-1}(e))$, where p is the natural projection $V \times \mathbb{R}^n \to \mathbb{R}^n$, hence it is continuous. The map g is an isomorphism on each fibre.
Now suppose $g' : E' \to E(\eta)$ is a bundle map, where $\pi' : E' \to B_1$ is a bundle and $\lambda g' = f\pi'$. We map $E' \to E_1$ by mapping

$$e' \mapsto (\pi'(e'), g'(e')) \in E_1.$$

Because g' is an isomorphism on each fibre, so is this map; and it induces the identity on the base space. Hence it is an equivalence.

$$\begin{array}{ccccc} E_1 & \xrightarrow{g} & E(\eta) & \xleftarrow{g'} & E' \\ \pi\downarrow & & \downarrow\lambda & & \downarrow\pi' \\ B_1 & \xrightarrow[f]{} & B(\eta) & \xleftarrow[f']{} & B_1 \end{array}$$

2.9. ***Definition.*** Let ξ, η be two bundles over B. The ***Whitney sum*** $\xi \oplus \eta$ is a bundle defined as the induced bundle $d^*(\xi \times \eta)$ for $d : B \to B \times B$ be the diagonal map and the product bundle $E(\xi) \times E(\eta) \to B \times B$.

$$\begin{array}{ccc} \xi \oplus \eta = d^*(\xi \times \eta) & \to & E(\xi) \times E(\eta) \\ \downarrow & & \downarrow \\ B & \xrightarrow[d]{} & B \times B \end{array}$$

The proof of the following is left as an exercise.

a) the fibre over b in $\xi \oplus \eta$ is $F_b(\xi) \times F_b(\eta)$, so that $\dim(\xi \oplus \eta) = \dim \xi + \dim \eta$,

b) $\oplus$ is commutative: $\xi \oplus \eta \simeq \eta \oplus \xi$,

c) $\oplus$ is associative: $(\xi \oplus \eta) \oplus \varsigma \simeq \xi \oplus (\eta \oplus \varsigma)$.

2.10. ***Definition.*** If ξ, η are bundles over B, then $g : E(\xi) \to E(\eta)$ is a ***homomorphism*** if

1) it maps each fibre linearly into a fibre,

2) the induced map on B is the identity.

Note that an equivalence is both a bundle map and a homomorphism. An ***embedding*** of bundles is a 1 - 1 homomorphism.

2.11. *Theorem.* *If $f : E(\xi) \to E(\eta)$ maps each fibre linearly into a fibre, then f may be factored into a homomorphism followed by a bundle map.*

***Proof*:** Let π_1, π_2 be the projections in ξ, η, respectively.
Let $f_B : B(\xi) \to B(\eta)$ be the map induced by f. Let $E_1 = f_B{}^*\eta$ be the bundle induced by f_B; let g be the bundle map $E_1 \to E(\eta)$ and π be the projection $E_1 \to B(\eta)$.

$$\begin{array}{ccccc} E(\xi) & \xrightarrow{h} & E_1 & \xrightarrow{g} & E(\eta) \\ \downarrow{\scriptstyle\pi_1} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi_2} \\ B(\xi) & \xrightarrow[i]{} & B(\xi) & \xrightarrow[f_B]{} & E(\eta) \end{array}$$

Define $h : E(\xi) \to B(\xi) \times E(\eta)$ by the equation $h(e) = (\pi_1(e), f(e))$. The image of h actually lies in that subset of $B(\xi) \times E(\eta)$ which is E_1; then h is a homomorphism. From the definition $f = gh$. □

2.12. *Lemma.* *Let ξ, η be bundles over B of dimensions n, p, respectively; let $g : \xi \to \eta$ be a homomorphism. If g is onto, then the kernel (g) is a bundle. If g is 1 - 1, then the cokernel (g), i.e., the quotient η / image (g), is a bundle.*

***Proof*:** Suppose g is 1 - 1 (i.e., has rank n when restricted to each fibre.) In $E(\eta)$, we define $e \sim e'$ if $e - e'$ exists and is in the image of g. We identify the elements of these equivalence classes; the resulting identification space is defined to be $E(\eta / g(\xi))$. It is a bundle over B with projection naturally defined and each fibre is a vector space of dimension $p - n$. We need only to show the existence of a local product structure.
Let U be an open set in B, with $\xi \mid U$ equivalent to $U \times \mathbb{R}^n$ and $\eta \mid U$ equivalent to $U \times \mathbb{R}^p$. Let g_0 denote the homomorphism of $U \times \mathbb{R}^n \to U \times \mathbb{R}^p$ induced by g. Now $(\eta / g(\xi)) \mid U$ is equivalent to the quotient $U \times \mathbb{R}^p / g_0(U \times \mathbb{R}^n)$, so that it suffices to show that this latter quotient is locally a product.
g_0 is given by a matrix $M(b) \in \mathcal{M}(p, n)$ which depends continuously on the point $b \in U$. Given b_0, we may assume that in a neighbourhood U_0 of b_0, the first n rows are independent. We define $h : U_0 \times \mathbb{R}^n \times \mathbb{R}^{p-n} \to U \times \mathbb{R}^p$ as the linear function on whose matrix (non-singular) is

$$\left[\begin{array}{c|c} M(b) & \begin{array}{c} 0 \\ \hline I_{p-n} \end{array} \end{array}\right]$$

The image of $U_0 \times \mathbb{R}^n \times 0$ under h is just $g_0(U_0 \times \mathbb{R}^n)$; since h is an equivalence, it induces an equivalence of

$$U_0 \times \mathbb{R}^{p-n} \simeq U_0 \times \mathbb{R}^n \times \mathbb{R}^{p-n} / U_0 \times \mathbb{R}^n \times 0 \ \text{ onto } \ U_0 \times \mathbb{R}^p / g_0(U_0 \times \mathbb{R}^n).$$

Secondly, suppose g is onto (i.e., it has rank p on each fibre.) $E(g^{-1}(0))$ is defined as that subset of $E(\xi)$ consisting of points e with $g(e) = 0$. Again, we need to show the existence of a local product structure. Let U, g_0, and $M(b)$ be as above. Given b_0, we may assume that the first p columns of are independent in the neighbourhood U_0 of b_0. We define $h : U_0 \times \mathbb{R}^n \to U_0 \times \mathbb{R}^p \times \mathbb{R}^{n-p}$ by the matrix function

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{M(b)} \\ \hline 0 & I_{p-n} \end{array}\right]$$

Now h followed by the natural projection of $U_0 \times \mathbb{R}^p \times \mathbb{R}^{n-p}$ onto $U_0 \times \mathbb{R}^p$ equals $g_0 \mid U$. Hence h^{-1} maps $U_0 \times 0 \times \mathbb{R}^{n-p}$ onto $g_0^{-1}(U_0 \times 0)$; since h is an equivalence, so is the restriction of h^{-1} to $U_0 \times 0 \times \mathbb{R}^{n-p}$. □

Remark. If g is onto, $\xi / g^{-1}(0)$ is a bundle, being the quotient of the inclusion homomorphism $g^{-1}(0) \to \xi$. If g is 1 - 1, $g(\xi)$ is a bundle, being the kernel of the projection homomorphism $\eta \to g(\xi)$.

2.13. ***Definition.*** If φ is a non-negative function on B, the ***support*** of φ is the closure of the set of x with $\varphi(x) > 0$. A ***partition of unity*** is a collection $\{\varphi_\alpha\}$ of non-negative functions on B, such that the sets $\{C_\alpha\} = \{\text{support}(\varphi_\alpha)\}$ form a locally-finite covering of B, and $\sum\varphi_\alpha(x) = 1$ (this is a finite sum for each x.)

2.14. ***Lemma.*** *Let B be a normal space; $\{U_\alpha\}$ a locally-finite open covering of B. Then there is a partition of unity $\{\varphi_\alpha\}$ with* support$(\varphi_\alpha) \subset U_\alpha$ *for each α.*

Proof: First, we show that there is an open covering $\{V_\alpha\}$ of B with $\overline{V}_\alpha \subset U_\alpha$ for each α. Assume that U_α are indexed by a set of ordinals (well-ordering theorem.) Let V_α be defined for all $\alpha < \beta$ and assume that the sets V_α along with the sets U_α for $\alpha \geq \beta$ cover B. Consider the set $A(\beta) = B \setminus \cup_{\alpha<\beta} V_\alpha \setminus \cup_{\alpha>\beta} U_\alpha$. Then $A(\beta) \subset U_\beta$. Let V_β be an open set containing the closed set $A(\beta)$, with $\overline{V}_\beta \subset U_\beta$ (normality.) This completes the construction of the V_α.

Now let g_α be a function which is positive on $\overline{V}_\alpha$ and 0 outside U_α (normality again.) Define $\varphi_{\alpha 0}(x) = g_{\alpha 0}(x) / \sum g_\alpha(x)$. Since $\{U_\alpha\}$ is locally-finite, the sum in the denominator is finite and positive, so $\{\varphi_\alpha\}$ is well-defined. □

Remark[1]***.*** If B is a differentiable manifold, φ_α may be chosen to be differentiable: Cover B with coordinate systems (V_i, h_i) as in 1.25 refining the covering U_α, $B \setminus \overline{V}_\alpha$. Let $\varphi_i(y) = \varphi_i(h_i(y))$ for $y \in V_i$, and $\varphi_i(y) = 0$ otherwise (φ as in 1.26.) Let $g_\alpha(y) = \sum\varphi_i(y)$, where the sum extends over all i such that $V_i \subset U_\alpha$.

2.15. ***Lemma.*** *Let B be paracompact and let $0 \to \xi \xrightarrow{i} \eta \xrightarrow{\varphi} \zeta \to 0$ be an exact sequence of homomorphism of bundles. Then there is equivalence $f : \eta \to \xi \oplus \zeta$, with fi the natural inclusion and φf^{-1} the natural projection.*

Proof: Let $\dim \xi = n$; $\dim \zeta = p$.
We first construct a Riemannian metric on η (i.e., a continuous inner product in $E(\eta)$.) Let $\{U_\alpha\}$ be a locally-finite covering of B with $\eta \mid U_\alpha$ trivial; let g_α be the corresponding projection of $\eta \mid U_\alpha$ onto $\mathbb{R}^{n+p}$. Let $\{\varphi_\alpha\}$ be a partition of unity with support$(\varphi_\alpha) \subset U_\alpha$.

If e, e' are in $E(\eta)$ and $\pi(e) = \pi(e')$, define $e \cdot e' = \sum_\alpha \varphi_\alpha(\pi(e)) g_\alpha(e) \cdot g_\alpha(e')$, where the dot on the right hand side is the ordinary scalar product in $\mathbb{R}^{n+p}$. This is a finite sum; it satisfies the axioms for a scalar product.
The way we use the Riemannian metric is to break η up into $iE(\xi)$ and its orthogonal complement. Let ξ' be the image of ξ in η and let $E(\zeta')$ be defined as that subset of consisting of elements which are orthogonal to $iE(\xi)$. In order to show that ζ' has a local product structure, consider the homomorphism

$$h : \eta \to \zeta'$$

which sends each vector into its orthogonal projections in ξ'. [Verification that h is continuous. Over any coordinate neighbourhood U we can choose a basis $a_1, \ldots a_n$ for the fibre of ξ'. Then the

1 See Appendix, Proposition A.

function h carries $v \in E(\eta)$ into $\sum t_j a_j \in E(\xi') \subset E(\eta)$, where $t_j = \sum B_{jk}(v \cdot a_k)$ and where (B_{jk}) denotes the inverse matrix to $(a_j \cdot a_k)$.] Since h is onto, its kernel ζ' is again a vector space bundle.

Now the bundle $i(\xi) = \xi'$ is equivalent to ξ. It remains to show that ξ' is equivalent to ζ and that η is equivalent to $\xi' \oplus \zeta'$. The former follows immediately from the fact that $\varphi \mid \zeta'$ is a homomorphism; form rank considerations it must be 1 - 1 and onto as well. The latter follows by noting that $E(\xi' \oplus \zeta')$ is defined as the subset of $E(\xi') \times E(\zeta')$ consisting of points (e_1, e_2) such that $\pi(e_1) = \pi(e_2)$. Consider the map f of $E(\xi' \oplus \zeta')$ into $E(\eta)$ obtained by taking (e_1, e_2) into their sum in $E(\eta)$ (their sum exists because e_1 and e_2 lie in the same fibre.) This is clearly a homomorphism; from rank considerations, it must be 1 - 1 and onto. □

2.16. *Definition*. Let M_1, M_2 be differentiable manifolds and let $f : M_1 \to M_2$ be an immersion. The ***normal bundle*** ν_f is defined as follows:

Let τ_1, τ_2 be the tangent bundles of M_1, M_2 respectively. By 2.11, the map $df : E(\tau_1) \to E(\tau_2)$ may be factored into a homomorphism h of $E(\tau_1)$ into $E(f^*\tau_2)$ followed by a bundle map g. Now h is a 1 - 1 homomorphism because f is an immersion; hence by 2.12, $f^*\tau_2$ / image (h) is a bundle over M_1. It is called the normal bundle ν_f.

Then $0 \to \tau_1 \to f^*\tau_2 \to \nu_f \to 0$ is an exact sequence if homomorphisms, so that by 2. 15, $f^*\tau_2$ is equivalent to $\tau_1 \oplus \nu_f$. Indeed, given a Riemannian metric on $f^*\tau_2$, ν_f is equivalent to the orthogonal complement of the image of τ_1.
Let us consider the case $M_2 = \mathbb{R}^{n+p}$, where dim $M_1 = n$. Then τ_2 is the trivial bundle, so that $f^*\tau_2$ is as well. (*Proof*: If $f : B_1 \to B(\eta)$ and η is trivial, so is $f^*\eta$. We have the diagram

$$\begin{array}{ccc} & & B \times \mathbb{R}^n \\ & & \downarrow \pi \\ f : B_1 & \longrightarrow & B \end{array}$$

$E(f^*\eta)$ is defined as that subset of $B_1 \times (B \times \mathbb{R}^n)$ consisting of points (b_1, b, x) such that $f(b_1) = (b, x)$, i.e., of all points $(b_1, f(b_1), x)$. If we map this into (b_1, x), we obtain an equivalence of $f^*\eta$ with the bundle $B_1 \times \mathbb{R}^n \to B_1$.
Thus $\tau_1 \oplus \nu_f$ is equivalent to a trivial bundle. In what follows, we investigate the following question: Given ξ, does there exist an η with $\xi \oplus \eta$ trivial? Using 1.28, this is always the case for ξ the tangent bundle of an n-manifold, and indeed η may be chosen also to have dimension n. A more general answer appears in 2.19.

2.17. *Definition*. Let $f : M_1{}^n \to M_2{}^p$; If f has rank p at every point of M_1, it is said to be ***regular***. If f is regular, the homomorphism $h : \tau_1 \to f^*\tau_2$ given by 2.11 is an onto map. By 2.12, the kernel of h is a bundle α_f. It is called the ***bundle along the fibre***.

Note that $f^{-1}(y)$ is a submanifold of M_1 of dimension $n - p$ (by 1.12 or 1.34.) The inclusion i_y of $f^{-1}(y)$ into M_1 induces an inclusion di_y of its tangent bundle into τ_1. The kernel of h consists precisely of the vectors which are in the image of some di_y, i.e., the vectors tangent to the submanifolds $f^{-1}(y)$ are the ones carried into 0 by h.

One has the exact sequence $0 \to \alpha_f \to \tau_1 \to f^*\tau_2 \xrightarrow{g} 0$, so that by 2.15, τ_1 is equivalent to $\alpha_f \oplus f^*\tau_2$.

2.18. *Definition*. A bundle ξ is of ***finite type*** if B is normal and may be covered by a finite number of neighbourhoods $U_1, \ldots U_k$ such that $\xi \mid U_i$ is trivial for each i.

2.19. *Lemma*. *ξ is of finite type if B is compact, or paracompact finite dimensional.*

Proof: The former statement is clear; let us consider the latter. By definition, the dimension of B is not greater than n if every covering has an open refinement such that

$$\text{no point of } B \text{ is contained in more than } n+1 \text{ elements of the refinement.} \tag{*}$$

It is a standard theorem of topology that an n-manifold has dimension n in this sense.
Cover B by open sets U, with $\xi \mid U$ trivial; let $\{V_\alpha\}$ be an open refinement of this covering satisfying (*). By 1.22, we may assume that $\{V_\alpha\}$ is locally-finite as well. Let $\{\varphi_\alpha\}$ be a partition of unity with support$(\varphi_\alpha) \subset V_\alpha$ for each α (2.14.)

Let A_i be the set of unordered $(i+1)$-tuple of distinct elements of the index set of $\{\varphi_\alpha\}$. Given a in A_i, where $a = \{\alpha_0, \ldots, \alpha_n\}$, let W_{ia} be the set of all x such that $\varphi_\alpha(x) < \min\{\varphi_{\alpha 0}(x), \ldots, \varphi_{\alpha n}(x)\}$ for all $\alpha \neq \alpha_1, \ldots, \alpha_i$. Each set W_{ia} is open, and $W_{ia} \cap W_{ib} = \varnothing$ if $a \neq b$. Also W_{ia} is contained in the intersection of the supports of $\varphi_{\alpha 0}(x), \ldots, \varphi_{\alpha i}(x)$, and hence in some set V_α. If we set X_i equal to the union of all sets W_{ia}, for fixed i, $\xi \mid X_i$ is trivial. Note that $\xi \mid W_{ia}$ is trivial and W_{ia} are disjoint.

Finally, the sets $X_0, \ldots, X_n$ cover B. Given x in B, x is contained in at most $n+1$ of the sets V_α, so that at most $n+1$ of the functions φ_α are positive at x. Since some φ_α is positive at x, x is contained in one of the sets W_{ia} for $0 \leq i \leq n$.
[The intuitive idea of the proof is as follows: Consider an n-dimensional simplicial complex, with φ_α the barycentric coordinate of x with respect to the vertex α. The sets W_{0a} will be disjoint neighbourhoods of the vertices, the sets W_{1a} disjoint neighbourhoods of the open 1-simplices, and so on.] □

2.20. *Theorem.* *If ξ is of finite type, there is a bundle η such that $\xi \oplus \eta$ is trivial.*

Proof: We proceed by showing that ξ may be embedded in a trivial bundle $B \times \mathbb{R}^m$, so that we have the exact sequence $0 \to \xi \xrightarrow{i} B \times \mathbb{R}^m \to B \times \mathbb{R}^m / i(\xi) \to 0$ by 2.12. The theorem then follows from 2.15. (Paracompactness is not needed since the trivial bundle clearly has a Riemannian metric.)
Cover B by finitely many neighbourhoods $U_1, \ldots, U_k$ with $\xi \mid U_i$ trivial for each i. Let $\varphi_1, \ldots, \varphi_k$ be a partition of unity with support$(\varphi_i) \subset U_i$ for each i (2.14). Let f_i denote the equivalence of $E(\xi \mid U_i)$ onto $U_i \times \mathbb{R}^n$; let $f_i^1, \ldots, f_i^n$ denote the coordinate functions of its projection into $\mathbb{R}^m$.
We define $h : E(\xi) \to B \times \mathbb{R}^{mk}$ as follows:

$$h(e) = (\pi(\mathrm{e}), (\varphi_i\,\pi(e))\cdot f_i^j(e)) \quad i = 1, \ \ldots, k; \quad j = 1, \ldots, n$$

(no summation indicated.) This is well-defined, since $\varphi_i\,\pi(e) = 0$ unless $e \in E(\xi \mid U_i)$. It is clearly a homomorphism, since each f_i^j is linear on $E(\xi \mid U_i)$. To show that it is 1 - 1, let $e \neq 0$. Then for some i, $\varphi_i\pi(e) > 0$. Since f_i is an equivalence, $f_i^j(e) \neq 0$ for some j. Hence $h(e) \neq (\pi(\mathrm{e}), 0)$ as desired. □

2.21. *Definition.* The bundle ξ is ***s-equivalent***[2] to η if there are trivial bundles o^p, o^n such that $\xi \oplus o^p \cong \eta \oplus o^n$.

Here $o^p = B \times \mathbb{R}^p$. Symmetry and reflexivity are clear. To show transitivity, assume $\xi \oplus o^p \cong \eta \oplus o^q$ and $\eta \oplus o^r \cong \zeta \oplus o^s$. Then $\xi \oplus o^p \oplus o^r \cong \zeta \oplus o^s \oplus o^q$.

Remark: s-equivalence differs from from equivalence. E.g., consider the two-sphere S^2 in $\mathbb{R}^3$. Then $\tau^2 \oplus \nu^1 \cong o^3$. The normal bundle ν^1 is easily seen to be trivial; but it is a classical theorem of topology that τ^2 is not (it does not admit a non-zero cross-section.) Hence τ^2 is s-trivial, but not trivial.

2 Short for "stably equivalent".

2.22. *Theorem.* *The set of s-equivalence classes of vector space bundles of finite type over B forms an abelian group under* $\oplus$[3].

***Proof*:** To avoid logical difficulties, we consider only subbundles of $B \times \mathbb{R}^m$, for all m. This suffices, since any bundle of finite type may be embedded in some $B \times \mathbb{R}^m$, by 2.20. The class o^p of trivial bundles is the identity element. The existence of inverses is the substance of 2.20. □

2.23. *Corollary.* *Given two immersions of the differentiable manifold M in euclidean space, their normal bundles are s-equivalent.* □

2.24. *Definition.* M^n is a ***π-manifold*** if M may be embedded in some $\mathbb{R}^{n+p}$ so that its normal bundle is trivial.

This is equivalent to the requirement that τ^n be s-trivial; Let τ^n be s-trivial. If we take some immersion of M into $\mathbb{R}^{n+p}$, then $\tau^n \oplus \nu^p$ is trivial by 2.16, so that ν^p is s-trivial, i.e., $\nu^p \oplus o^q = o^{p+q}$ for some q. Consider the composite immersion $M \to \mathbb{R}^{n+p} \subset \mathbb{R}^{n+p+q}$. The normal bundle of M in $\mathbb{R}^{n+p+q}$ is just $\nu^p \oplus o^q$, which is trivial.
Conversely, if ν^p is trivial for some immersion, then τ^n is s-trivial because $\tau^n \oplus \nu^p$ is trivial.

2.25. *Definition.* Let $G_{p,n}$ denote the set of all n-dimensional vector subspaces of $\mathbb{R}^{n+p}$ (i.e., all n-dimensional hyperplanes through the origin.) It is called the ***Grassman manifold*** of n-planes in $n+p$ space.
Its topology is obtained as follows; Consider $\mathcal{M}(n, n+p; n)$; we identify two elementss of this set if the hyperplane spanned by their row vectors are the same. $G_{p,n}$ is in 1 - 1 correspondence with this identification space, and is given the identification topology. Let ρ be the projection

$$\rho : \mathcal{M}(n, n+p; n) \to G_{p,n}.$$

Now $\rho(A) = \rho(B)$ if and only if $A = CB$ for some non-singular $n \times n$ matrix C: The hyperplane $\rho(A)$ consists of all points $(x^1, \ldots, x^{n+p})$ $\mathbb{R}^{n+p}$ which equal $(c^1, \ldots, c^n)\cdot A$ for some choice of constants c^i. If $\rho(A) = \rho(B)$, then

$$\begin{aligned} (1, 0, \ldots, 0)\cdot A &= (c^1{}_1, \ldots, c^n{}_1)\cdot B \\ (0, 1, \ldots, 0)\cdot A &= (c^1{}_2, \ldots, c^n{}_2)\cdot B \\ \cdots &= \cdots \\ (0, 0, \ldots, 1)\cdot A &= (c^1{}_n, \ldots, c^n{}_n)\cdot B \end{aligned}$$

for some choice of $c^j{}_i$. Then $IA = CB$, where C has rank n because A does. The converse is clear.

(a) $G_{p,n}$ is locally euclidean. Let $A \in \mathcal{M}(n, n+p; n)$; after permuting the columns, we may assume $A = (P, Q)$ where P is $n \times n$ and non-singular. Let U be the set of all such A; it is an open set in $\mathcal{M}(n, n+p; n)$, being the inverse image of the non-zero reals under the continuous map $(P, Q) \to \det P$. If $\rho(P, Q) = \rho(R, S)$, where P is non-singular, then $(P, Q) = (CR, CS)$ for some non-singular C. Hence R is necessarily non-singular; it follows that $\rho^{-1}(\rho(U)) = U$, so that $\rho(U)$ is open in $G_{p,n}$ (by definition of the identification topology.)

We show $\rho(U)$ homeomorphic with $\mathbb{R}^{pn}$. Define $\varphi : U \to \mathbb{R}^{pn}$ by $\varphi(P, Q) = P^{-1}Q$. If $\rho(P, Q) = \rho(R, S)$

3 The resulting abelian group is called the K-group of B. For more on this, see "*Vector Bundles and K-Theory*" by Allen Hatcher in his homepage http://www.math.cornell.edu/~hatcher/#ATI.

then $(P, Q) = (CR, CS)$, so that

$$P^{-1}Q = (CR)^{-1}(CS) = R^{-1}S.$$

Hence φ induces a continuous map $\varphi_0 : \rho(U) \to \mathbb{R}^{pn}$. Define $\psi : \mathbb{R}^{pn} \to \rho(U)$ by $\psi(Q) = \rho(I, Q)$ where Q is an $n \times p$ matrix. One checks immediately that ψ and φ_0 are inverse of each other.

$$\begin{tikzcd}
\mathcal{M}(n, n+p; n) \arrow[d, "\rho"] & \supset & U \arrow[d, "\rho"] \arrow[dr, "\varphi"] & \\
G_{p,n} & \supset & \rho(U) \arrow[r, shift left, "\varphi_0"] & \mathbb{R}^{pn} \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

(b) To show that $G_{p,n}$ is Hausdorff, we show that maps every compact set into a closed set (this will clearly suffice.) Let K be a compact subset of $\mathbb{R}^{pn}$; we show $\varphi^{-1}(K)$ is closed in $\mathcal{M}(n, n+p; n)$. $\varphi^{-1}(K)$ consists of all matrices (P, Q) with P non-singular and $P^{-1}Q \in K$. Let $(P, Q) \in \mathcal{M}(n, n+p; n)$ be the limit of the sequence $\{(P_i, Q_i)\}$ of elements of $\varphi^{-1}(K)$. Since K is compact, some subsequence of the sequence $\{\varphi(P_i, Q_i)\} = \{P_i^{-1}Q_i\}$ converges to a point R of K. Then the corresponding subsequence of the sequence $\{Q_i\}$ converges to PR, so that $C = P(I, R)$. Since (P, Q) has rank n it follows that P is non-singular, so that $(P, Q) \in \varphi^{-1}(K)$, as desired.
Hence $G_{p,n}$ is a manifold of dimension pn.

(c) $G_{p,n}$ is a differentiable manifold and ρ is a differentiable map. A function f on the open set V in $G_{p,n}$ belongs to the differentiable structure $\mathcal{D}$ if $f\rho$ is differentiable. To show that this satisfies the condition for a differentiable structure, we show that $(\rho(U), \varphi_0)$, as defined in (a), is a coordinate system. Let f be defined on $V \subset \rho(U)$. Given $Q \in \mathbb{R}^{pn}$, $f\varphi_0^{-1}(Q) = f\rho(I, Q)$ so that $f\varphi_0^{-1}$ is differentiable if $f\rho$ is. Conversely, given $(P, Q) \in V$, $f\rho(P, Q) = f\varphi_0^{-1}\varphi_0\rho(P, Q) = f\varphi_0^{-1}(P^{-1}Q)$, so that $f\rho$ is differentiable if $f\varphi_0^{-1}$ is.

(d) $G_{p,n}$ is compact. Let L be the subset of $\mathcal{M}(n, n+p; n)$ consisting of matrices whose rows are orthonormal vectors. L is a closed and bounded subset of $\mathbb{R}^{n(n+p)}$. Since $\rho(L) = G_{p,n}$ (the Gram-Schmidt orthogonalisation process proves this), $G_{p,n}$ is compact.

(e) $G_{p,n}$ is diffeomorphic to $G_{n,p}$. Geometrically, the homeomorphism h is defined as carrying each hyperplane into its orthogonal complement. It is clearly 1 - 1; to show it is differentiable we use the coordinate system $(\rho(U), \varphi_0)$ defined in (a). Let g map U into $\mathcal{M}(n, n+p; n)$ by carrying (P, Q) into $(-(P^{-1}Q)^\tau, I_p)$; it is differentiable (τ denotes transpose.) The row space of (P, Q) is the same as that of $(I_n, P^{-1}Q)$, while the row vectors of this matrix are orthogonal to those of $(-(P^{-1}Q)^\tau, I_p)$ (multiply the one by the transpose of the other.) Hence g induces $h \mid \rho(U)$, so that the latter is differentiable.

2.26. ***Definition.*** Let $E(\gamma_p{}^n)$ be defined as that subsets of $G_{p,n} \times \mathbb{R}^{n+p}$ consisting of pairs (H, x) where x is a vector lying in the hyperplane H. It is called the ***universal bundl***e (for reasons we shall see.) The projection π maps (H, x) into H; the fibre is thus an n-dimensional subspace of $\mathbb{R}^{n+p}$.

$\gamma_p{}^n$ is an n-dimensional vector space bundle over $G_{p,n}$. We need to show the existence of a local product structure. Let $(\rho(U), \varphi_0)$ be a coordinate neighbourhood on $G_{p,n}$, as in (a) above. We define $h : \rho(U) \times \mathbb{R}^n \to \pi^{-1}\rho(U)$ as carrying $(H, (x^1, \ldots, x^n))$ into $(x^1, \ldots, x^n)\cdot(I_n, Q)$ where $Q = \varphi_0(H)$. This is a vector in the hyperplane H; h is clearly an isomorphism on each fibre. Its inverse is continuous, since it sends $(H, (y^1, \ldots, y^{n+p}))$ in $G_{p,n} \times \mathbb{R}^{n+p}$ into $(H, (y^1, \ldots, y^n))$ in $\rho(U) \times \mathbb{R}^n$.

2.27. *Definition.* ξ is a ***differentiable vector space bundle*** if $E(\xi)$ and $B(\xi)$ are differentiable manifolds, and if the homeomorphisms

$$U \times \mathbb{R}^n \to \pi^{-1}(U)$$

which specify the local product structure can be chosen as diffeomorphisms.

It follows that $\pi : E \to B$ is differentiable of maximum rank. Note that B can be differentiably embedded in E by mapping b into the 0-vector of F_b. The normal bundle of this embedding is just ξ.
Examples of differentiable bundles include the tangent bundles of a manifold, the normal bundle of an immersed manifold, and the universal bundle $\gamma_p{}^n$ above. In the latter case, $E(\gamma_p{}^n)$ is embedded differentiably in $G_{p,\,n} \times \mathbb{R}^{n+p}$.

2.28. *Theorem.* *Let ξ^n be an n-dimensional vector space bundle. The following conditions are equivalent*:

(a) *ξ is of finite type.*

(b) *There is a bundle η^p such that $\xi^n \oplus \eta^p$ is trivial.*

(c) *There is a bundle map $\xi^n \to \gamma_p{}^n$ for some p.* (*Thus the terminology* "*universal bundle*" *for $\gamma_p{}^n$.*)

Proof: We have already shown that (a) $\Longrightarrow$ (b) (2.20); the bundle η^p there constructed has dimension $n(k-1)$, where k is the number of elements in the covering $U_1, \ldots, U_k$ of $B(\xi) = B$ such that $\xi \mid U_i$ is trivial.

(b) $\Longrightarrow$ (c): Condition (b) means that ξ^n may be embedded in the trivial bundle $B(\xi) \times \mathbb{R}^{n+p}$; let f be this embedding. We wish to define g and g_B in the following diagram:

$$\begin{array}{ccc} E(\xi) & \xrightarrow{g} & E(\gamma_p{}^n) \\ \pi\downarrow & & \downarrow \\ B(\xi) & \xrightarrow[g_B]{} & G_{p,\,n} \end{array}$$

Since f is a 1 - 1 homomorphism, $f(F_b)$ is the cartesian product of b and an n-dimensional hyperplane H^n in $\mathbb{R}^{n+p}$; let $g_B(b) \equiv H^n$. If $e \in F_b$, then $f(e) = (b, x)$ where x is a vector in the hyperplane H^n; let $g(e) = (H^n, x)$ in $G_{p,\,n} \times \mathbb{R}^{n+p}$. Then $g(e)$ actually lies in the subset of $G_{p,\,n} \times \mathbb{R}^{n+p}$ which constitutes $E(\gamma_p{}^n)$. From rank considerations, g is automatically an isomorphism on each fibre.

It remains to show that g is continuous. Locally, g just looks like a map $U \times \mathbb{R}^n \to G_{p,\,n} \times \mathbb{R}^{n+p}$. We factor it into a continuous map $h : U \times \mathbb{R}^n \to \mathcal{M}(n, n+p;\, n) \times \mathbb{R}^{n+p}$ followed by the projection $\rho \times 1$ into $G_{p,\,n} \times \mathbb{R}^{n+p}$. Locally, f looks like a map $U \times \mathbb{R}^n \to B \times \mathbb{R}^{n+p}$. Let $e_1, \ldots, e_n$ be a basis for $\mathbb{R}^n$; we define $h(b, x)$ as $(A, p_2 f(b, x))$. Here p_2 projects $B \times \mathbb{R}^{n+p}$ onto its second factor and A is the matrix having $p_2 f(b, e_1), \ldots, p_2 f(b, e_n)$ as its rows. Then h is continuous and $(\rho \times 1)h$ equals g. (Note: The converse assertion , (c) implies (b), can be proved by the same argument.)

(c) $\Longrightarrow$ (a): Being compact, $G_{p,\,n}$ is covered by a finitely many neighbourhoods U_i with $\gamma_p{}^n \mid U_i$ trivial. (In fact, $(n+p)!\,/\,n!p!$ neighbourhoods will suffice.) If f is a bundle map $\xi^n \to \gamma_p{}^n$ then the sets $\{f_B{}^{-1}(U_i) = V_i\}$ cover B, and $\xi \mid V_i$ is equivalent to the bundle induced by $f_B : V_i \to G_{p,\,n}$ (the uniqueness part of 2.8.) Then $\xi \mid V_i$ is trivial (since it is induced from a trivial bundle.) □

Chapter III *The Cobordism Theory of Thom*

3.1. *Definition*. An ***n-manifold with boundary*** Q is a Hausdorff space with a countable basis which is locally homeomorphic with $\mathbb{H}^n$ (the subset of $\mathbb{R}^n$ such that $x^1 \geq 0$.) The ***boundary*** ∂Q is that subset of corresponding to $\mathbb{R}^{n-1}$ under the local homeomorphism ($\mathbb{R}^{n-1}$ being the subset of $\mathbb{R}^n$ with $x^1 = 0$.) ∂Q is well-defined, since the image of an open set in $\mathbb{R}^n$ under a homeomorphism of it into $\mathbb{R}^n$ must be open (Brouwer theorem on invariance of domain.) It is clear that ∂Q is an $(n-1)$-manifold.
A ***differential structure*** $\mathcal{D}$ on Q is a collection of real-valued functions f defined on open subsets of Q such that

1) Every point of Q has an open neighbourhood U and a homeomorphism h of U into an open subset of $\mathbb{H}^n$, such that f is in $\mathcal{D}$ if and only if fh^{-1} is differentiable. (f is defined on an open subset of U; fh^{-1} differentiable means that it may be extended to a neighbourhood of $h(U)$ ***in*** $\mathbb{R}^n$ so as to be differentiable.)

2) If U_i are open sets contained in the domain of f and $U = \cup U_i$, then $f \mid U \in \mathcal{D}$ if and only if $f \mid U_i \in \mathcal{D}$ for each i.

As before, $(U, h)$is called a ***coordinate system*** on Q, and one can define differentiable structure alternatively by means of coordinate systems.

We impose an additional condition on $\mathcal{D}$ in 3.2.

3.2. *Definition*. Let M_1, M_2 be compact differentiable n-manifolds. They are said to be in the same ***cobordism class*** ($M_1 \sim M_2$) if there is a compact differentiable $n + 1$ manifold-with-boundary Q such that ∂Q is diffeomorphic with the disjoint union of M_1 and M_2 (denoted by $M_1 + M_2$.)

Symmetry and reflexivity of this relation are clear. To show transitivity, we impose the additional condition on $\mathcal{D}$ that there is a neighbourhood U of ∂Q in Q which is diffeomorphic with $\partial Q \times [0, 1)$, the diffeomorphism being the identity on $\partial Q \times 0$. This is redundant, but we assume it to avoid proving it[4]. Transitivity follows:
Let $M_1 + M_2$ be diffeomorphic with ∂Q_1 and $M_2 + M_3$ be diffeomorphic with ∂Q_2; let h_1, h_2 be the diffeomorphisms. We form a new space Q_3 from $Q_1 \cup Q_2$ by identifying each point of $h_1(M_2)$ with its image under $h_2 h_1^{-1}$. There is then a homeomorphism of $M_2 \times (-1, 1)$ into this space which equals h_1 when restricted to $M_2 \times 0$, and is a diffeomorphism of $M_2 \times [0, (-1)^i)$ into Q_i for $i = 1, 2$. (It is derived from the postulated "product neighbourhoods" $\partial Q_i \times [0, 1)$.) If this is taken to be a coordinate system on Q_3, Q_3 becomes a differentiable manifold-with-boundary, and $M_1 + M_3$ is diffeomorphic with ∂Q_3. Q_1 and Q_2 diffeomorphic with subsets of Q_3.

3.3. *Definition*. As usual, there are logical difficulties involved in considering these cobordism classes. One way of avoiding them is to consider only manifolds-with-boundary embedded in some euclidean space $\mathbb{R}^n$: If Q_1 is a differentiable manifold-with-boundary and $Q_2 = \partial Q_1 \times [0, 1)$, then the space Q_3 constructed in the preceding paragraph is a differentiable manifold, so that it may be embedded in some euclidean space. Hence Q_1 may so be embedded.

With these restrictions, the set of cobordism classes of n-manifolds forms an abelian group (denoted

4 This fact is called the ***smooth collaring theorem***. See Appendix, Proposition B for a proof.

by $\mathcal{N}^n$) under the operation + (disjoint union.) If $M_1 \sim M_1'$ and $M_2 \sim M_2'$, this means that $M_i + M_i'$ is diffeomorphic with ∂Q_i. Then $(M_1 + M_2) + (M_1' + M_2')$ is diffeomorphic with $\partial(Q_1 \cup Q_2)$, so that $M_1 + M_2 \sim M_1' + M_2'$ and the operation + is well-defined on cobordism classes. The zero element is the vacuous manifold or the n-sphere (or ∂Q, where Q is any compact differentiable $(n + 1)$-manifold-with-boundary.) The remaining axioms are clear. Note that $M + M$ is diffeomorphic with $\partial(M \times [0, 1])$, so that every element is of order 2.

The groups $\mathcal{N}^n$ are called the (non-orientable) ***cobordism groups***. Let $\mathcal{N}$ denote the direct sum $\mathcal{N}^0 \oplus \mathcal{N}^1 \oplus \mathcal{N}^2 \oplus \cdots$. There is a bilinear symmetric pairing of $\mathcal{N}^i$, $\mathcal{N}^j$ into $\mathcal{N}^{i+j}$, i.e., a homomorphism of $\mathcal{N}^i \otimes \mathcal{N}^j$ into $\mathcal{N}^{i+j}$ induced by the operation of cartesian product.
First, $(M_1 + M_2) \times M_3 = (M_1 \times M_3) + (M_2 \times M_3)$ by definition of cartesian product. Second, if $M_1 \sim 0$, i.e., $M_1 = \partial Q$, then $M_1 + M_2$ is diffeomorphic with $\partial(Q \times M_2)$, so that $M_1 + M_2 \sim 0$.
Since $M_1 + M_2 \sim M_2 + M_1$, and since $M_1 \times p \sim M_1$ (where p is a point-manifold), this pairing makes $\mathcal{N}$ into a (graded) commutative ring with unit. Indeed, it is a graded algebra over the field $\mathbb{Z}/2\mathbb{Z}$.

3.4. ***Remark.*** The general result of Thom is the following

Theorem. *$\mathcal{N}$ is a polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ with one generator in each positive dimension except those of the form $2^m - 1$. If n is even, projective n-space is a generator.*

This theorem means that there are compact manifolds $M^2, M^4, M^5, \ldots$ such that every compact manifold is in the cobordism class of a disjoint union of products of these manifolds, and that there are no relations among the generators (except commutativity and associativity of products.)
Thom's procedure is to show that $\mathcal{N}^n$ is isomorphic with the $(n + k)^{th}$ homotopy group of a certain space T_k, and then to compute these homotopy groups. We shall consider only the first of these two problems in the present notes.

3.5. ***Definition.*** Let h be an embedding of the differentiable manifold M^n in $\mathbb{R}^{n+k}$; consider the normal bundle of this embedding. Using the standard Riemannian metric for the tangent bundle to $\mathbb{R}^{n+k}$, this normal bundle is equivalent to the orthogonal complement of the image in the tangent bundle of $\mathbb{R}^{n+k}$ of the tangent bundle of M^n (2.16); this complement we denote by ν^k. Define e as the canonical map of $E(\nu^k)$ into $\mathbb{R}^{n+k}$ which maps the vector v normal to at x into its end point. (Described differently, one maps the tangent bundle to $\mathbb{R}^{n+k}$ into itself canonically by mapping the vector v, based at x, into the point $v + x$ of $\mathbb{R}^{n+k}$. This map is differentiable; its restriction to $E(\nu^k)$ is the map e.)
Consider M^n as the zero vectors of $E(\nu^k)$. Then we have the

3.6. ***Theorem.*** *There is a neighbourhood of M^n in $E(\nu^k)$ which is mapped diffeomorphically onto a neighbourhood of M^n in $\mathbb{R}^{n+k}$.*

***Proof*:** Note that e is differentiable, and that it has rank $n + k$ at points of $M^n \subset E(\nu^k)$. (This is easily checked by computing the derivative matrix of e with respect to a local coordinate system.) Hence e has rank $n + k$ in some neighbourhood of M^n in $E(\nu^k)$, so that it is a local homeomorphism at points of M^n: It maps a neighbourhood of each $x \in M^n$ homeomorphically onto a neighbourhood of $e(x)$. We then appeal to the topological

Lemma. *Let X, Y be Hausdorff spaces with countable bases and X be locally compact. If $f : X \to Y$ is a local homeomorphism and the restriction of f to the closed subset A is a homeomorphism, then f is a homeomorphism on some neighbourhood V of A.*

This lemma is proved as follows:

1) If A is compact, the lemma holds. For otherwise, there would be points x, y arbitrarily close to A such that $f(x) = f(y)$. Since A has a compact neighbourhood, we may choose sequences $\{x_n\}$, $\{y_n\}$ converging to x, y respectively, in A such that $x_n \neq y_n$ and $f(x_n) = f(y_n)$. Hence $f(x) = f(y)$ so that $x = y$, f being a homeomorphism on A. But then f is not a local homeomorphism at x.
2) Let A_0 be a compact subset of A. Then there is a neighbourhood U_0 of A_0 such that $\overline{U}_0$ is compact and f is a homeomorphism on $\overline{U}_0 \cup A_0$: It will suffice for f to be 1 - 1, since f is a local homeomorphism. By (1), let V_0 be a neighbourhood of A_0 so that is $f \mid \overline{V}_0$ 1 - 1. If no neighbourhood of A_0 in V_0 satisfies the requirement for U_0, there is a sequence $\{x_n\}$of $X \setminus A$ converging to $x \in A_0$ with $f(x_n) \in f(A)$. Choose $y_n \in A$ with $f(x_n) = f(y_n)$. Since f is continuous, $\{f(y_n)\}$ converges to $f(x)$; since f is a homeomorphism on A, $\{y_n\}$converges to x. Since $x_n \neq y_n$, this contradicts the fact that f is a local homeomorphism at x.
3) Express A as the union of an ascending sequence of compact sets $A_1 \subset A_2 \subset \cdots$. Let V_1 be a neighbourhood of A_1 such that $\overline{V}_1$ is compact and f is a homeomorphism on $\overline{V}_1 \cup A$ (by (2).) Given V_i a neighbourhood of A_i satisfying these conditions, consider the set $\overline{V}_i \cup A_{i+1}$. It is a compact subset of $\overline{V}_i \cup A$, and f is a homeomorphism on $\overline{V}_i \cup A$. Hence by (2) there is a neighbourhood V_{i+1} of $\overline{V}_i \cup A_{i+1}$ with $\overline{V}_{i+1}$ compact, such that f is a homeomorphism on $\overline{V}_i \cup A_{i+1}$. We proceed by induction: f is 1 - 1 on $V = \cup V_{i+1}$, so that it is a homeomorphism on V (being a local homeomorhism-onto.) □

3.7. *Corollary.* *Any differentiable submanifold of* $\mathbb{R}^{n+k}$ *is a differentiable neighbourhood retract.*

Proof: The projection of $E(\nu^k) \to M^n$ induces (under e) a differentiable map of a neighbourhood of M^n in $\mathbb{R}^{n+k}$ onto M^n which is the identity on M^n. □

3.8. *Definition.* Let ξ be a vector space bundle with $B(\xi)$ compact; Let $T(\xi)$ denote the 1-point compactification of $E(\xi)$. It is called the ***Thom space*** of ξ. Let ∞ denote the added point.

Let ξ have a Riemannian metric. Let $T_\varepsilon(\xi)$ be obtained from $E(\xi)$ by identifying all vectors of length greater than or equal to ε to a point. Let $\alpha(x)$ be a C^∞ function with $\alpha'(x) \geq 0$ which equals 1 in a neighbourhood of $x = 0$ and $\to \infty$ as $x \to 1$. The map of $E(\xi)$ into $T(\xi)$ which carries the vector e into the vector $e\alpha(||e|| / \varepsilon)$ induces a homeomorphism of $T_\varepsilon(\xi)$ onto $T(\xi)$ which is a diffeomorphism on the set $E_\varepsilon(\xi)$, consisting of vectors of length less than ε. The fact that B is compact is used here.

3.9. *Definition.* Let the compact manifold M^n be embedded in $\mathbb{R}^{n+k}$. ν^k is given the Riemannian metric of $\mathbb{R}^{n+k}$; by 3.6 there is a neighbourhood of M^n in $\mathbb{R}^{n+k}$ which is diffeomorphic to the subset $E_{2\varepsilon}(\nu^k)$ of $E(\nu^k)$. Such a neighbourhood is called a ***tubular neighbourhood*** of M^n.

By 3.8, we see that $T(\nu^k)$ is homeomorphic with the space obtained from $\mathbb{R}^{n+k}$ by collapsing the exterior of the ε-neighbourhood of M^n to a point.

We will need three lemmas concerning approximation by differentiable functions.

3.10. *Lemma.* *Let A be a closed subset of the differentiable manifold* M^n, *let* $f : M^n \to \mathbb{R}^m$ *be differentiable on A. Let* δ *be a positive continuous function on* M^n. *There exists* $g : M^n \to \mathbb{R}^m$ *such that*

1) *g is differentiable*,
2) *g is a* δ*-approximation to f*,

3) $g \mid A = f \mid A$.

Proof: It suffices to prove this lemma in the case $m = 1$.
Given $x \in A$, $f \mid A$ may be extended to a differentiable function f_x in a neighbourhood N_x of x. Let N_x be chosen small enough that $|f_x(y) - f(y)| < \delta(y)$ for all $y \in N_x$.
Given $x \in M^n \setminus A$, choose a neighbourhood N_x of x small enough that $|f(y) - f(x)| < \delta(y)$ for all $y \in N_x$. Define $f_x(y) \equiv f(x)$ for $y \in N_x$.
Let $\{\varphi_\alpha\}$ be a differentiable partition of unity with support(φ_α) contained in some N_x, say $N_{x(\alpha)}$, for each α. Define $g(y) = \sum_\alpha \varphi_\alpha(y) f_{x(\alpha)}(y)$. One checks the conditions of the lemma easily. □

More generally:

3.11. *Lemma.* *Let $f : M_1 \to M_2$ be a continuous map of differentiable manifolds which is differentiable on the closed subset A of M_1. Let $\varepsilon(x) > 0$ be given; and give M_2 the metric determined by some embedding $M_2 \subset \mathbb{R}^p$. Then there exists a differentiable map $g : M_1 \to M_2$ such that*

1) *g is differentiable,*
2) *g is an ε-approximation to f,*
3) *$g \mid A = f \mid A$.*

Proof: There is a neighbourhood U of M_2 in $\mathbb{R}^p$ of which is a differentiable retract (3.7.) Let ρ be the differentiable retraction of U onto M_2. Let $\delta(x)$ be a positive function on M_2 so chosen that the cubical neighbourhood of $f(x)$ of radius $\delta(x)$ lies in U, and so that its image under ρ has radius less than $\varepsilon(x)$. Let $f_1 : M_1 \to \mathbb{R}^p$ be a differentiable map which is a δ-approximation to f, such that $f_1 \mid A = f \mid A$ (by 3.10.) Define $g(x) = \rho(f_1(x))$. □

3.12. *Lemma.* *Let $f : M_1 \to M_2$ be a continuous map of differentiable manifolds; let the metric on M_2 be obtained by embedding it in some euclidean space. Given $\varepsilon(x)$, there is a $\delta(x)$ such that if $g : M_1 \to M_2$ is a δ-approximation to f, g is homotopic to f under a homotopy $F(x, t)$ with*

1) *$F(x, t) = f(x)$ for any x such that $g(x) = f(x)$ and*
2) *$F(x, t)$ is a an ε-approximation to f for any t.*

Proof: Let U, ρ, and $\delta(x)$ be chosen as in 3.11. Let $g : M_1 \to M_2$ be a δ-approximation to f. Then the line segment from $g(x)$ to $f(x)$ lies in U, so that

$$F(x, t) = \rho(tg(x) + (1 - t)f(x))$$

is well defined. Furthermore $F(x, t)$ is an ε-approximation to $f(x)$ for any t. □

3.13. *Definition*. We wish to define a homomorphism $\lambda : \pi_{n+k}(T(\xi^k), \infty) \to \mathcal{N}^n$ where $\mathcal{N}^n$ is the cobordism class of the base space for $T(\xi^k)$. To this end we need some preparation:

Let ξ^k be a differentiable vector space bundle with $B(\xi)$ compact and m-dimensional; let $E(\xi^k)$ be given a metric by embedding it as a closed differentiable submanifold in some euclidean space (it is an $(m + k)$-manifold.)
Given an element of $\pi_{n+k}(T(\xi^k), \infty)$, let it be represented by the map

$$f : (\overline{C}_{n+k}, \partial\overline{C}_{n+k}) \to (T(\xi^k), \infty),$$

where $\overline{C}_{n+k}$ is the closed cube $[0, 1]^{n+k}$ and $\partial\overline{C}_{n+k}$ is the boundary. Let U denote the open subset

$f^{-1}(E(\xi^k))$ of C_{n+k}. Let $g : U \to E(\xi^k)$ be a differentiable δ-approximation to $f \mid U$, where δ is so chosen that $\delta < 1$ and g is homotopic to f, the homotopy F also being a 1-approximation to f. (This ensures that F will be continuous if we define $F(x, t) = \infty$ for $x \in \overline{C}_{n+k} \setminus U$.)

Now g may be approximated in turn by a differentiable map $h : U \to E(\xi^k)$ which is transverse regular on the submanifold $B(\xi^k)$ of $E(\xi^k)$. We choose the approximation close enough to h, the homotopy H being a 1-approximation to g for each t. Extend h to $\overline{C}_{n+k}$ by defining $h(x) = \infty$ for $x \in \overline{C}_{n+k} \setminus U$. Then h is in the homotopy class of f.
$h^{-1}(B(\xi^k))$ is a differentiable submanifold M^n of U which is closed in $\overline{C}_{n+k}$, and thus compact.

3.14. *Theorem.* *Define* $\lambda : \pi_{n+k}(T(\xi^k), \infty) \to \mathcal{N}^n$ *by assigning* the cobordism class $[M^n] \in \mathcal{N}^n$ *to the homotopy class* $[h] \in \pi_{n+k}(T(\xi^k), \infty)$. *Then* λ *is a well-defined homomorphism.*

***Proof*:** Let $H : (\overline{C}_{n+k} \times I, \partial\overline{C}_{n+k} \times I) \to (T(\xi^k), \infty)$ be a homotopy between $h_0 = H(x, 0)$ and $h_1 = H(x, 1)$. Let h_0, h_1 satisfy the conditions

1) h_i is differentiable on $h_i^{-1}(E(\xi^k))$
2) h_i is transverse regular on $B(\xi^k)$. ($i = 0, 1$.)

We wish to show that $h_0^{-1}(B)$ and $h_1^{-1}(B)$ belong to the same cobordism class.
We may assume that $H(x, t) = H(x, 0)$ for $t \leq 1/3$, and $H(x, t) = H(x, 1)$ for $t \geq 2/3$. Let $U = H^{-1}(E(\xi^k)) \cap [\overline{C}_{n+k} \times (0, 1)]$; then U is an open subset of $\mathbb{R}^{n+k+1}$. Let $G : U \to E(\xi^k)$ be a differentiable 1-approximation to H which equals H on the closed subset A, where $A = U \cap [\overline{C}_{n+k} \times (0, ¼] \cup [¾, 1)]$. (See 3.11. H is differentiable on A.)

Now G satisfies the transverse regularity condition for $B(\xi^k)$ at points in A (since h_0 and h_1 are transverse regular on $B(\xi^k)$) so that by 1.35 there is a differentiable map $F : U \to E(\xi^k)$ which equals G on A, is transverse regular on $B(\xi^k)$, and is a 1-approximation to G. Because F is a 2-approximation to H, it remains continuous if we define $F(x, t) = \infty$ for $(x, t) \in (\overline{C}_{n+k} \times (0, 1)) \setminus U$. Because F equals H on A, it remains continuous if we define $F(x, t) = H(x, t)$ for $t = 0, 1$. Hence $F^{-1}(B)$ is a compact subset of $\overline{C}_{n+k}$, being closed and bounded.

Because $F \mid U$ is transverse regular on B, $(F \mid U)^{-1}(B)$ is a differentiable $(n + 1)$-submanifold of $\overline{C}_{n+k} \times (0, 1)$. Then

$$(F \mid U)^{-1}(B) \cap \overline{C}_{n+k} \times t = \begin{array}{ll} h_0^{-1}(B) \times t & \text{for } t \in [0, ¼], \\ h_1^{-1}(B) \times t & \text{for } t \in [¾, 1]. \end{array}$$

Hence $F^{-1}(B)$ is a differentiable manifold-with-boundary whose boundary is $h_0^{-1}(B) + h_1^{-1}(B)$. Thus λ is well-defined.
It is trivial to show λ is a homomorphism, because the sum in $\mathcal{N}^n$ is derived from disjoint union of representative manifolds. □

3.15. *Theorem.* *If* ξ^k *is the universal bundle* γ_m^k *where* $k \geq n + 1$, $m \geq n$ *then* $\lambda : \pi_{n+k}(T(\xi^k), \infty) \to \mathcal{N}^n$ *is onto.*

***Proof*:** Let M^n be a compact manifold; let $k \geq n + 1$. Let M^n be embedded in C_{n+k} (1.32); let ν^k be the normal bundle of this embedding. The Riemannian metric on $E(\nu^k)$ is that derived from the natural scalar product on the tangent bundle to $\mathbb{R}^{n+k}$, in which ν^k is contained.
By 3.6, for small ε the subset of $E_{2\varepsilon}(\nu^k)$ of $E(\nu^k)$ is diffeomorphic with a tubular neighbourhood of M^n in C_{n+k}; let U be the image of $E_\varepsilon(\nu^k)$.

Let p_1 project $\overline{C}_{n+k}$ onto the space obtained from $\overline{C}_{n+k}$ by identifying $\overline{C}_{n+k} \setminus U$ to a point (denoted by $\overline{C}_{n+k} / (\overline{C}_{n+k} \setminus U)$).
Let p_2 be the diffeomorphism of U onto $E_\varepsilon(\nu^k)$, followed by the map of $E(\nu^k)$ into $T_\varepsilon(\nu^k)$ which identifies all vectors of length $\geq \varepsilon$ (3.8.) p_2 is then extended by mapping $\overline{C}_{n+k} \setminus U$ into ∞.
Let p_3 be the homeomorphism of $T_\varepsilon(\nu^k)$ onto $T(\nu^k)$ constructed in 3.8. The composite map $p_3p_2p_1$ is a diffeomorphism of U onto $E(\nu^k)$.
Finally, let p_4 be the bundle map of ν^k into $\gamma_m{}^k$ induced from the embedding of M^n in $\mathbb{R}^{n+k} \subset \mathbb{R}^{m+k}$.
Because both fibres have dimension k, this map satisfies the transverse regularity condition for $G_{k,m}$ at each point of M^n. Extend p_4 in the obvious way to map $T(\nu^k)$ into $T(\gamma_m{}^k)$.

Let $g = p_4p_3p_2p_1$. Then $g : \partial\overline{C} \to \infty$. Let $\mu(M^n)$ denote the homotopy class of g in $\pi_{n+k}(T(\xi^k), \infty)$. Now g is transverse regular on $G_{k,m}$ and $M^n = g^{-1}(G_{k,m})$. By definition, the cobordism class of M^n is the image of $\mu(M^n)$ under λ, so that $\lambda\mu(M^n) = [M^n]$. □

3.16. *Theorem*. *If ξ^k is the universal bundle $\gamma_m{}^k$ where $k \geq n + 2$, $m > n$ then λ is one-to-one.*

Proof: Given an element of $\pi_{n+k}(T(\xi^k), \infty)$, we may suppose it represented by a map

$$f : (\overline{C}_{n+k}, \partial\overline{C}_{n+k}) \to (T(\xi^k), \infty)$$

which is differentiable on $f^{-1}(E)$ and transverse regular on $G_{m,k}$ (by 3.13.) Let $M^n = f^{-1}(G_{m,k})$; we wish to show that if M^n is the boundary of an $(n+1)$-manifold-with-boundary Q, then f is homotopic to the constant map.

M^n is a submanifold of C_{n+k}; let its normal bundle be ν^k. Let ε be chosen so that $E_{2\varepsilon}(\nu^k)$ is diffeomorphic with the 2ε-neighbourhood of M^n; let U_ε be the image of the vectors of $E_\varepsilon(\nu^k)$. Impose a Riemannian metric on $\gamma_m{}^k$; let δ be so chosen that $\|x\| \geq \varepsilon$ implies $\|f(x)\| \geq \delta$ for $x \in E(\nu^k)$.

***Step* 1**. f is homotopic to a map f_1 such that

1) f_1 is differentiable on $f_1{}^{-1}(E)$ and transverse regular on $G_{m,k}$.
2) $f = f_1$ on $M^n = f^{-1}(G_{m,k})$.
3) f_1 carries everything outside U_ε into ∞.

Define $F : E(\gamma_m{}^k) \to T(\gamma_n{}^k)$ by the equation $F(e, t) = e\alpha(t\|e\| / \delta)$, where α is the function defined in 3.8. Let $f_1(x) = F(f(x), 1)$.

***Step* 2**. By the diffeomorphism of $U_{2\varepsilon}$ with $E_{2\varepsilon}$, f_1 induces a map $\bar{f}_1$ of $\overline{E}_\varepsilon(\nu^k)$ into $T(\gamma_n{}^k)$ which carries $\partial(E_\varepsilon)$ into ∞. Any homotopy of $\bar{f}_1$ which leaves $\partial(E_\varepsilon)$ at ∞ induces a homotopy of f_1.
Now $\bar{f}_1$ is homotopic to a map $\bar{f}_2$ such that

1) $\bar{f}_2$ is differentiable on $\bar{f}_2{}^{-1}(E)$ and transverse regular on $G_{m,k}$.
2) $\bar{f}_2 = \bar{f}_1$ on $M^n = f^{-1}(G_{m,k})$.
3) $\bar{f}_2$ *is* locally a bundle map in some neighbourhood of M^n.

The homotopy leaves $\partial(E_\varepsilon)$ at ∞.

Consider $G : \overline{E}(\gamma_m{}^k) \times I \to T(\gamma_n{}^k)$ defined by the equation $G(e, t) = \bar{f}_1(te) / t$. As $t \to 0$, $G(e, t)$ approaches a limit which is non-zero if $e \neq 0$ (since $\bar{f}_1$ is differentiable and transverse regular.) It is easily seen to be a bundle map. It will not suffice for our purpose, since it does not carry $\partial(E_\varepsilon) \times I$ into ∞. Choose $\delta > 0$ so that $\|x\| \geq \varepsilon$ implies $\|G(x, t)\| \geq \delta$ for $x \in E(\nu^k)$, $t \in I$, and define

$$H(e, t) = [G(e, t)]\alpha(-\|G(e, t)\| / \delta).$$

If we set $\bar{f}_2 = H(e, 0)$, then $\bar{f}_2$ is a bundle map for $\|e\|$ small (since $\alpha(x) = 1$ for x small.) The map $H(e, 1) = \bar{f}_1(e)\alpha(\|\bar{f}_1(e)\| / \delta)$ does not equal $\bar{f}_1$, but it is homotopic to $\bar{f}_1$, the homotopy leaving $\partial(E_\varepsilon)$ at ∞. The homotopy is defined by the equation

$$K(e, t) = \bar{f}_1(e)\alpha(t\|\bar{f}_1(e)\| / \delta), \text{ as in Step 1.}$$

***Step* 3**. Let Q be the $n + 1$ manifold-with-boundary such that $M^n = \partial Q$. Let h be a diffeomorphism of $M^n \times [0, 1]$ into Q which carries $M^n \times 0$ onto ∂Q.
Define $h_1 : Q \to C_{n+k} \times I$ as follows:

$h_1(x) = h(y, t)$ if $x = h(y, t)$ where $(y, t) \in (M^n, [0, ½])$.
$h_1(x) = p$, where p is some fixed point interior to $C_{n+k} \times I$ if $x \notin$ image h.
$h_1(x) = (1- \beta(t))h(y, ½) + \beta(t)p$, where β is a C^∞ function with $\beta'(t) \geq 0$, $\beta(t) = 0$ in a neighbourhood of $t = ½$ and $\beta(t) = 1$ in a neighbourhood of $t = 1$ if $x = h(y, t)$ where $(y, t) \in (M^n, [½, 1])$.

h_1 is a differentiable map of Int Q into Int$(C_{n+k} \times I)$; and h_1 is a 1 - 1 immersion in a neighbourhood of ∂Q. Since $\dim(C_{n+k} \times I) > 2(n + 1)$, h_1 may be approximated by a 1 - 1 immersion h_2 which equals h_1 in a neighbourhood of ∂Q (by 1.29.) It may be extended to an embedding of Q into $C_{n+k} \times I$. (Since Q is compact, a 1 - 1 immersion is automatically an embedding.) Let Q now be considered as this subset of $C_{n+k} \times I$.

***Step* 4**. We have a map f_2 of $\bar{C}_{n+k} \times 0$ into $T(\gamma_n{}^k)$ which is a bundle map when restricted to a small tubular neighbourhood of $M^n \times 0$ in $C_{n+k} \times 0$. We extend it to $\bar{C}_{n+k} \times [0, b)$ for b small in a trivial way. Suppose there exists a map g of the ε'-neighbourhood N of Q in $C_{n+k} \times I$ into $T(\gamma_n{}^k)$ which equals f_2 in some neighbourhood of ∂Q in $C_{n+k} \times I$ and maps each point of $N \setminus Q$ into a non-zero vector in $E(\gamma_n{}^k)$. Our theorem then follows: Let δ be so chosen that, if the distance$(x, Q) \geq \varepsilon'/2$, then $\|g(x)\| \geq \delta$.

Define $g_1 : C_{n+k} \times I \to T(\gamma_n{}^k)$ by the equation

$$g_1(x, s) = \begin{cases} g(x, \varepsilon)\alpha(\|g(x, s)\| / \delta) & \text{for } (x, s) \in N, \text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

The restriction of g_1 to $C_{n+k} \times 0$ does not equal the map f_2, but it is homotopic to f_2, by the same technique as used at the end of Step 2. g_1 is the homotopy required for our theorem.

To show that the extension g exists, we refer to Steenrod, "Fibre Bundles" (Princeton University Press, 1951.) According to §19.4 and §19.7 of this book, the principal bundle associated with $\gamma_n{}^k$ is an m-universal bundle. That is: given a vector space bundle ξ^k over a complex of dimension $\leq m$, any bundle map of ξ^k, restricted to a subcomplex, into $\gamma_n{}^k$ can be extended throughout ξ^k. We will assume the well known result that Q can be triangulated. The dimension $n + 1$ of Q is $\leq m$. Hence any bundle map of the normal bundle ν^k of Q, restricted to a polyhedral neighbourhood of ∂Q, into $\gamma_n{}^k$ can be extended throughout ν^k.

Applying this result to the map f_2, this completes the proof of 3.16. □

Letting T_k stand for the union of the Thom spaces $T(\gamma_n{}^k) \subset T(\gamma_{n+1}{}^k) \subset \cdots$, in the weak topology, Theorem 3.15 and 3.16 imply the following.

3.17. *Theorem.* *The cobordism group* $\mathcal{N}^n$ *is canonically isomorphic to the stable homotopy group* $\pi_{n+k}(T_k)$, *for* $k \geq n + 2$. □

References

Morse, A. P.: *The behavior of a function on its critical set*, Annals of Mathematics Volume **40** (1939), 62-70.

Sard, A.: *The measure of the critical values of differentiable maps*, Bulletin of the American Mathematical Society Volume **48** (1942), 883-890.

Thom, R.: *Quelques propriétés globales des variétés différentiables*, Commentarii Mathematici Helvetici Volume **28** (1954), 17-86.

Whitney, H.: *A function not constant on a connected critical set of points*, Duke Mathematical Journal, Volume **1** (1935), 514-517.

--------------: *Differentiable manifolds*, Annals of Mathematics **37** (1936), 645-680.

--------------: *The self-intersections of a smooth n-manifold in* 2*n-space*, Annals of Mathematics Volume **45** (1944), 220-246.

--------------: *The singularities of a smooth n-manifold in* $(2n - 1)$ *space*, Annals of Mathematics Volume **45** (1944), 247-293.

Appendix[5]

In this appendix we give a proof for the smooth collaring theorem. Our exposition follows Dirk Schütz. (See "Lecture06_handout.pdf" in the "material" for MAGIC002, in "courses" listed in the page "http://maths.dept.shef.ac.uk/magic/courses.php".)

First we show that partitions of unity allow us to glue together smooth functions which are only defined on subsets of a differentiable manifold M.

Proposition A: *Let* $\{U_\alpha\}$ *be an open cover of the differentiable manifold M and* $\{\varphi_\alpha\}$ *a partition of unity with* support$(\varphi_\alpha) \subset U_\alpha$. *For every* α, *assume that* $f_\alpha : U_\alpha \to \mathbb{R}^k$ *is a smooth function. Then* $f : M \to \mathbb{R}^k$ defined by

$$f(x) = \textstyle\sum_\alpha f_\alpha(x)\varphi_\alpha(x)$$

is a well defined smooth function.

Proof: Observe that $f_\alpha \cdot \varphi_\alpha : U_\alpha \to \mathbb{R}^k$ has support contained in support(φ_α), so can be extended to a smooth function on M. Also, by the local finiteness, the formula for f is locally just a finite sum, so smoothness follows. □

The same procedure can be used to extend vector fields defined on each V_i to a vector field on M.

Proposition B (***Smooth Collaring Theorem***): *Let M be a compact differentiable manifold with boundary. Then there exists an embedding* $i : \partial M \times [0, 1) \to M$ *with* $i(x, 0) = x$ *for all* $x \in \partial M$.

Proof: Let $U_1, \ldots, U_k$ be a finite covering of M by coordinate charts, and let $\{\varphi_i : U_i \to [0, 1]\}$ be a partition of unity subordinate to this cover.
Case I: U_i is diffeomorphic to an open set of $\mathbb{R}^n$. Define a vector field v_i on U_i to be identically zero.
Case II: U_i contains boundary points. Let $\varphi_i : U_i \to U_i'$ be a chart, and define a vector field v_i on U_i such that the induced vector field on $U_i' \subset \mathbb{H}^n$ is constant $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$.

We get a vector field on M by using the partition of unity. Let Φ be the corresponding flow. As M is compact, and since the vector field is chosen on the boundary so that it is not possible to flow "out" of the manifold, we get a smooth flow

$$\Phi : M \times [0, \infty) \to M$$

It is easy to check that $\Phi \mid \partial M \times [0, 1)$ is the desired embedding. □

5 Added by the transcriber.